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ABSTRACT

The method adopted by the UK National Physical Laboratory for calibrating underwater acoustic transducers in the frequency range 10 kHz to 500 kHz is three-transducer spherical-wave reciprocity. This report is concerned with the evaluation of measurement uncertainty associated with estimates of the transmitting and receiving sensitivities of a device provided by NPL's implementation of this calibration method. Consideration is given to formulating the problem of uncertainty evaluation in terms of a measurement model, which is developed from a sequence of sub-models that describe different aspects of the measurement, and the available information about the quantities involved in the model. A comparison is made of the results obtained using conventional 'uncertainty propagation' and a Monte Carlo method as approaches to solving the formulated problem. The sensitivity of the results obtained to assumptions about the correlations associated with influence quantities in the calibration and the reliability of the available information about those quantities is investigated.

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Contents

1 Introduction

The most common absolute methods used for the calibration of underwater acoustic transducers rely on the principle of acoustic reciprocity, which, when applied to free-field spherical wave reciprocity calibrations, states that the transmitting and receiving sensitivities of a device are related by the so-called reciprocity parameter, depending only on frequency and the density of water. There are various ways of implementing reciprocity-based calibration methods, but they all involve the measurement of the current driving a transducer and the voltage generated by another transducer placed in the acoustic field produced by the first transducer. The method adopted by the UK National Physical Laboratory (NPL) for calibrating underwater acoustic transducers in the frequency range $10\,\mathrm{kHz}$ to $500\,\mathrm{kHz}$ is *three-transducer spherical-wave reciprocity* [8]. In this free-field method, three transducers are paired in three measurement configurations and the above electrical measurements are made. Given knowledge of the geometry of the acoustic field propagating between the two devices (in this case, a spherical wave geometry), and assuming that at least one device is reciprocal, the transmitting and receiving sensitivities of any of the three devices can be determined from the electrical measurements.

This report is concerned with the evaluation of measurement uncertainty associated with the estimates of transmitting and receiving sensitivities of a device provided by NPL's implementation of the method of three-transducer spherical-wave reciprocity. The aims of the work are summarized as follows:

1. To give a formulation of the problem of uncertainty evaluation in terms of (a) a measurement model that defines the mathematical relationship between all quantities involved in the measurement, and (b) the available information about those quantities;

2. To compare the results obtained from different approaches to solving the formulated problem and, in particular, to validate the results obtained using the conventional approach of 'uncertainty propagation' with those obtained from a Monte Carlo method that provides a more generally-applicable approach;

3. To investigate the impact on the results of different assumptions about the correlations associated with input quantities in the measurement model used to describe corrections applied as part of the calibration. In practice, it can be difficult to quantify such correlations, and an uncertainty analysis can be undertaken for different assumptions in order to investigate the sensitivity of the results to those assumptions;

4. To investigate the sensitivity of the results to different assumptions about the 'reliability' of the available information about the input quantities in the measurement model.

The overall objective, therefore, is to give confidence in the reported uncertainties that accompany results obtained from the calibration method.

The report is organized as follows. The method of free-field reciprocity for calibrating an underwater acoustic transducer is described in section 2. The problem of uncertainty evaluation for the calibration of an underwater acoustic transducer is formulated in section 3, including a description of the measurement model that defines the mathematical relationship between all quantities involved in the measurement. An approach to treating correlation associated with input quantities in the measurement model is also presented. Approaches to solving the problem of uncertainty evaluation, including the GUM uncertainty framework and a Monte Carlo method and a comparison of those approaches, are described in section 4. Results obtained for a problem of uncertainty evaluation that is indicative of a calibration of an underwater acoustic transducer are given in section 5. Finally, conclusions are given in section 6.

2 Reciprocity method

In the reciprocity method three measurements are made, each involving two (from three) transducers, with one transducer acting as a transmitting device and the other as a receiving device. One transducer (labelled P) is used only as a projector or transmitting device, and has unknown transmitting response S_{P} to current. A second transducer (labelled H) is used only as a hydrophone or receiving device, and has unknown free-field receive sensitivity M_{H}. A third transducer (labelled T) is used both as a transmitting and receiving device, and is assumed to be a reciprocal device with unknown free-field receive sensitivity M_{T} and unknown transmitting response S_{T} to current.

The reciprocity method is based on using

1. the three measurements to provide an estimate of the product of M_{T} and S_{T} for the tranducer T, and
2. the *reciprocity principle* to provide an estimate of the quotient of M_{T} and S_{T} given by the free-field reciprocity parameter J.

In terms of the available information (the three measurements and the reciprocity of T), the method provides estimates of $S_{\mathrm{P}},\ M_{\mathrm{H}},\ M_{\mathrm{T}}$ and S_{T}.

Suppose a projector P with transmitting response S_{P} is driven by an alternating current of amplitude I_{P} and frequency f. Then, at a distance d from the reference centre of P and in the reference direction specified in the definition of S_{P}, the projector generates a sound pressure p given by

$$p = \frac{S_{\mathrm{P}} I_{\mathrm{P}}}{d},$$

where spherical spreading of the sound energy from the projector is assumed.

For the first measurement, the hydrophone H is placed in the sound field of the projector P so that its reference centre is a distance d_{PH} from the reference centre of P. The amplitude

V_{PH} of the open circuit voltage produced by H is

$$V_{\mathrm{PH}} = p\, M_{\mathrm{H}} = \frac{M_{\mathrm{H}} S_{\mathrm{P}} I_{\mathrm{P}}}{d_{\mathrm{PH}}},$$

and the transfer impedance Z_{PH} of the transducer pair P and H is given by

$$Z_{\mathrm{PH}} = \frac{V_{\mathrm{PH}}}{I_{\mathrm{P}}} = \frac{M_{\mathrm{H}} S_{\mathrm{P}}}{d_{\mathrm{PH}}}. \tag{1}$$

Here, as well as for the measurements described below, the distance between the projector and the hydrophone must be large enough to ensure that (a) the hydrophone is in the far field of the projector, and (b) the hydrophone receives a substantially plane wave.

For the second measurement, the transfer impedance Z_{PT} of the transducer pair P and T is determined after replacing the hydrophone H by the reciprocal transducer T, giving

$$Z_{\mathrm{PT}} = \frac{V_{\mathrm{PT}}}{I_{\mathrm{P}}} = \frac{M_{\mathrm{T}} S_{\mathrm{P}}}{d_{\mathrm{PT}}}, \tag{2}$$

where d_{PT} is the distance between the reference centres of P and T.

For the third measurement, the transfer impedance Z_{TH} for the transducer pair T and H is determined. In this case, the transducer T is used as a projector, and H again as a hydrophone, giving

$$Z_{\mathrm{TH}} = \frac{V_{\mathrm{TH}}}{I_{\mathrm{T}}} = \frac{M_{\mathrm{H}} S_{\mathrm{T}}}{d_{\mathrm{TH}}}, \tag{3}$$

where d_{TH} is the distance between the reference centres of T and H.

It follows from expressions (1) to (3) that

$$\frac{Z_{\mathrm{PT}} Z_{\mathrm{TH}}}{Z_{\mathrm{PH}}} = M_{\mathrm{T}} S_{\mathrm{T}} \frac{d_{\mathrm{PH}}}{d_{\mathrm{PT}} d_{\mathrm{TH}}}, \tag{4}$$

which involves the product of M_{T} and S_{T}.

For the reciprocal transducer T, the quotient of M_{T} and S_{T} is equal to the free-field reciprocity parameter $J = 2d_0/(\rho f)$ at the reference distance $d_0 = 1\,\mathrm{m}$, i.e.,

$$\frac{M_{\mathrm{T}}}{S_{\mathrm{T}}} = \frac{2}{\rho f}, \tag{5}$$

where ρ is the density of water.

From expressions (4) and (5), it follows that

$$M_{\mathrm{T}}^2 = \frac{2}{\rho f} \frac{d_{\mathrm{PT}} d_{\mathrm{TH}}}{d_{\mathrm{PH}}} \frac{Z_{\mathrm{PT}} Z_{\mathrm{TH}}}{Z_{\mathrm{PH}}}, \tag{6}$$

and

$$S_{\mathrm{T}}^2 = \frac{\rho f}{2} \frac{d_{\mathrm{PT}} d_{\mathrm{TH}}}{d_{\mathrm{PH}}} \frac{Z_{\mathrm{PT}} Z_{\mathrm{TH}}}{Z_{\mathrm{PH}}}. \tag{7}$$

Then, from expressions (3) and (7),

$$M_{\mathrm{H}}^2 = \frac{2}{\rho f}\frac{d_{\mathrm{PH}}d_{\mathrm{TH}}}{d_{\mathrm{PT}}}\frac{Z_{\mathrm{PH}}Z_{\mathrm{TH}}}{Z_{\mathrm{PT}}}, \tag{8}$$

and from expressions (2) and (6),

$$S_{\mathrm{P}}^2 = \frac{\rho f}{2}\frac{d_{\mathrm{PH}}d_{\mathrm{PT}}}{d_{\mathrm{TH}}}\frac{Z_{\mathrm{PH}}Z_{\mathrm{PT}}}{Z_{\mathrm{TH}}}. \tag{9}$$

Expressions (6) to (9) define the required quantities $S_{\mathrm{P}},\ M_{\mathrm{H}},\ M_{\mathrm{T}}$ and S_{T} in terms of the distances d between the devices and transfer impedances Z corresponding to the three measurements.

3 Measurement model

In this section are identified (a) the measurand or output quantity, the quantity intended to be measured, (b) the influence or input quantities on which the output quantity depends, and (c) the measurement model that defines the mathematical relationship between the output quantity and the input quantities. The available information about the input quantities is also indicated. The focus is on the determination of the free-field sensitivity M_{H} of the hydrophone H given by expression (8) for a single frequency f. Similar considerations would apply to the determinations of $S_{\mathrm{P}},\ M_{\mathrm{T}}$ and S_{T}.

The measurement model for the calibration of an underwater acoustic transducer by the method of free-field reciprocity is sufficiently complicated that it is described in terms of a sequence of sub-models relating to the following measurement and evaluation stages:

1. Measure voltages corresponding to drive currents;
2. Evaluate currents driving projectors;
3. Measure voltages from receivers;
4. Evaluate transfer impedances;
5. Evaluate hydrophone sensitivity.

Each sub-model is described below. A schematic diagram of the arrangement of the instrumentation in a calibration is shown in figure 1.

3.1 Measure voltages corresponding to drive currents

The quantities E_{PH}, E_{PT} and E_{TH} representing the voltages provided by the attenuator for the three configurations of the devices are given by

$$E_{\mathrm{PH}} = \widehat{E}_{\mathrm{PH}}G\delta E_{\mathrm{PH}}, \quad E_{\mathrm{PT}} = \widehat{E}_{\mathrm{PT}}G\delta E_{\mathrm{PT}}, \quad E_{\mathrm{TH}} = \widehat{E}_{\mathrm{TH}}G\delta E_{\mathrm{TH}}, \tag{10}$$

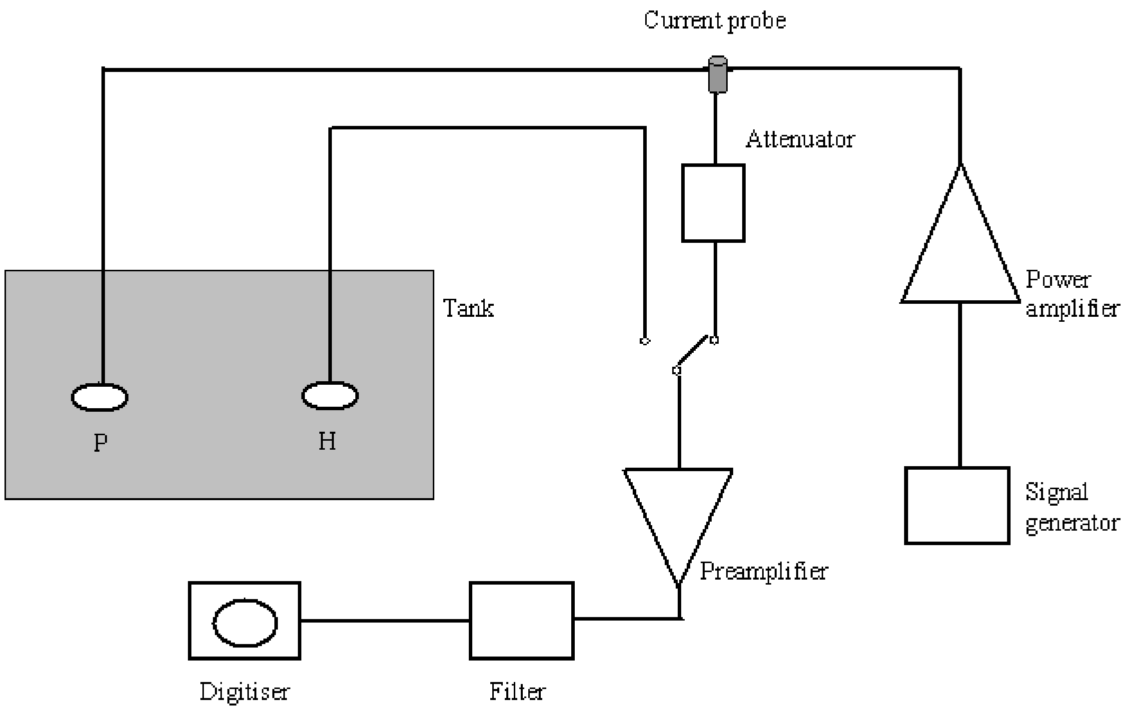

Figure 1: Schematic diagram showing the arrangement of the instrumentation in a calibration. P is the projector or transmitting device and H is the receiving device.

where

1. $\widehat{E}_{\rm PH}$, $\widehat{E}_{\rm PT}$ and $\widehat{E}_{\rm TH}$ are the measured values of the voltages,
2. G is an unknown systematic error quantity due to the instrumentation in the system, including preamplifier, filter, digitizer, etc., and
3. $\delta E_{\rm PH}$, $\delta E_{\rm PT}$ and $\delta E_{\rm TH}$ are independent random error quantities describing noise due to the instrumentation.

Each measured voltage value is obtained as the average of n values obtained independently (typically, $n = 4$) and the corresponding random error quantity has an expectation (estimate) of unity. The relative standard deviation (relative standard uncertainty associated with the estimate) of the random error quantity is assigned on the basis of a history of measured values obtained using the instrumentation. Each random error quantity is characterized by a Gaussian distribution.

3.2 Evaluate currents driving projectors

The quantities $I_{\rm PH}$, $I_{\rm PT}$ and $I_{\rm TH}$ representing the currents that drive the projectors in the three configurations of the devices are given by

$$I_{\rm PH} = E_{\rm PH} C^{\rm F} C^{\rm lin}_{\rm PH} A_{\rm PH}, \quad I_{\rm PT} = E_{\rm PT} C^{\rm F} C^{\rm lin}_{\rm PT} A_{\rm PT}, \quad I_{\rm TH} = E_{\rm TH} C^{\rm F} C^{\rm lin}_{\rm TH} A_{\rm TH}, \quad (11)$$

where

1. $C^{\rm F}$ is the calibration factor (the same for each current) for the current probe at the frequency of interest with measured value $\widehat{C}^{\rm F}$,
2. $C^{\rm lin}_{\rm PH}$, $C^{\rm lin}_{\rm PT}$ and $C^{\rm lin}_{\rm TH}$ are independent quantities (different for each current) describing corrections for the departure from linearity of the current probe, and
3. $A_{\rm PH}$, $A_{\rm PT}$ and $A_{\rm TH}$ are attenuation factors (different for each current) with values $\widehat{A}_{\rm PH}$, $\widehat{A}_{\rm PT}$ and $\widehat{A}_{\rm TH}$ set independently.

The calibration certificate for the current probe reports values for the calibration factor, and the associated uncertainties, at a number of different frequencies. The measured value $\widehat{C}^{\rm F}$ is obtained by applying a correction to a value reported on the certificate, e.g., the applied correction might be based on interpolating the reported values to the frequency of interest. The calibration factor at the frequency of interest is expressed as

$$C^{\rm F} = \widehat{C}^{\rm F} C^{\rm cal} C^{\rm cor}, \quad (12)$$

where $C^{\rm cal}$ is a quantity that describes the deviation of the (uncorrected) calibration factor reported on the calibration certificate from its measured value, and $C^{\rm cor}$ is a quantity that describes the correction of the reported calibration factor to the frequency of interest. The estimate of $C^{\rm cal}$ is unity, and the quantity is characterized by a Gaussian distribution, being based on information provided on a calibration certificate. The estimate of $C^{\rm cor}$ is also unity, and the quantity is characterized by a rectangular distribution with limits defining the maximum correction that is made to a reported calibration factor.

The quantities $C_{\rm PH}^{\rm lin}$, $C_{\rm PT}^{\rm lin}$ and $C_{\rm TH}^{\rm lin}$ describe the departure from linearity of the response of the current probe. The estimates of these quantities are unity, and the quantities are characterized by rectangular distributions with limits assigned on the basis of expert knowledge.

The attenuation factors are expressed as

$$A_{\rm PH} = \widehat{A}_{\rm PH}\delta A_{\rm PH}, \quad A_{\rm PT} = \widehat{A}_{\rm PT}\delta A_{\rm PT}, \quad A_{\rm TH} = \widehat{A}_{\rm TH}\delta A_{\rm TH}, \tag{13}$$

where $\delta A_{\rm PH}$, $\delta A_{\rm PT}$ and $\delta A_{\rm TH}$ are quantities that describe the deviations of the factors from the values to which they are set. The quantities have estimates of unity, and are characterized by Gaussian distributions, being based on information provided on the calibration certificate for the attenuator.

3.3 Measure voltages from receivers

The quantities $V_{\rm PH}$, $V_{\rm PT}$ and $V_{\rm TH}$ representing the voltages provided by the receivers in the three configurations of the devices are given by

$$\begin{aligned} V_{\rm PH} &= \widehat{V}_{\rm PH} G \delta V_{\rm PH} K_{\rm PH}^{\rm sp} K_{\rm PH}^{\rm ss} K_{\rm PH}^{\rm load} K_{\rm PH}^{\rm mis}, \\ V_{\rm PT} &= \widehat{V}_{\rm PT} G \delta V_{\rm PT} K_{\rm PT}^{\rm sp} K_{\rm PT}^{\rm ss} K_{\rm PT}^{\rm load} K_{\rm PT}^{\rm mis}, \\ V_{\rm TH} &= \widehat{V}_{\rm TH} G \delta V_{\rm TH} K_{\rm TH}^{\rm sp} K_{\rm TH}^{\rm ss} K_{\rm TH}^{\rm load} K_{\rm TH}^{\rm mis}, \end{aligned} \tag{14}$$

where

1. $\widehat{V}_{\rm PH}$, $\widehat{V}_{\rm PT}$ and $\widehat{V}_{\rm TH}$ are the measured values of the voltages,
2. G is an unknown systematic error quantity due to the instrumentation in the system, including preamplifier, filter, digitizer, etc., and is identical to that in expressions (10),
3. $\delta V_{\rm PH}$, $\delta V_{\rm PT}$ and $\delta V_{\rm TH}$ are independent random error quantities describing noise due to the instrumentation,
4. $K_{\rm PH}^{\rm sp}$, $K_{\rm PT}^{\rm sp}$ and $K_{\rm TH}^{\rm sp}$ are correlated quantities representing corrections due to lack of spherical spreading,
5. $K_{\rm PH}^{\rm ss}$, $K_{\rm PT}^{\rm ss}$ and $K_{\rm TH}^{\rm ss}$ are correlated quantities representing corrections due to lack of steady state,
6. $K_{\rm PH}^{\rm load}$, $K_{\rm PT}^{\rm load}$ and $K_{\rm TH}^{\rm load}$ are correlated quantities representing corrections for the electrical loading of the instrumentation in the system, including cables, preamplifier, etc., and

7. $K_{\mathrm{PH}}^{\mathrm{mis}}$, $K_{\mathrm{PT}}^{\mathrm{mis}}$ and $K_{\mathrm{TH}}^{\mathrm{mis}}$ are independent quantities representing corrections due to misalignment of the devices.

As in section 3.1, each measured voltage value is obtained as the average of n values obtained independently (typically, $n = 4$). The corresponding random error quantity has an expectation of unity, a relative standard deviation assigned on the basis of a history of measured values, and is characterized by a Gaussian distribution.

The calibration method assumes that the acoustic waves spread spherically. The quantities $K_{\mathrm{PH}}^{\mathrm{sp}}$, $K_{\mathrm{PT}}^{\mathrm{sp}}$ and $K_{\mathrm{TH}}^{\mathrm{sp}}$ describe corrections for lack of spherical spreading in the case that the separations between the transmitting and receiving devices in the three configurations are not sufficient for acoustic far-field conditions to hold. The estimates of the quantities are unity (so no correction to the measured voltage values is applied) but an uncertainty is associated with each estimate to describe incomplete knowledge of the value of the correction. The correlation associated with each pair of corrections is expected to be appreciable. An approach to treating the correlation is described in section 3.7.

The calibration method also assumes that the devices (which are resonant) have reached steady-state. To observe steady-state it is necessary to wait long enough for the transient behaviour of the devices to disappear, but not too long that the recorded signals are contaminated by the effects of reflections from the boundaries of the tank in which the calibration is undertaken. The corrections $K_{\mathrm{PH}}^{\mathrm{ss}}$, $K_{\mathrm{PT}}^{\mathrm{ss}}$ and $K_{\mathrm{TH}}^{\mathrm{ss}}$ made for this effect depend most strongly on the properties of the transmitting device, so the correlation associated with $K_{\mathrm{PH}}^{\mathrm{ss}}$ and $K_{\mathrm{PT}}^{\mathrm{ss}}$ can be expected to be strong, but that associated with $K_{\mathrm{PH}}^{\mathrm{ss}}$ (or $K_{\mathrm{PT}}^{\mathrm{ss}}$) and $K_{\mathrm{TH}}^{\mathrm{ss}}$ to be weak.

The electrical impedances of the devices are used to make corrections to the measured voltage values to account for the presence of extension cable and electrical loading from the pre-amplifier. The quantities $K_{\mathrm{PH}}^{\mathrm{load}}$, $K_{\mathrm{PT}}^{\mathrm{load}}$ and $K_{\mathrm{TH}}^{\mathrm{load}}$ describe these corrections, and are expected to be correlated because of the use of the same cables and pre-amplifier for the three measurements.

Finally, the devices are aligned either visually (making use of an alignment mark) or acoustically (by looking for a signal maximum). Visual alignment is used for nominally 'omnidirectional' devices with elements made from small spheres or cylinders, whereas acoustic alignment is used for directional transducers such as 'piston' devices. The corrections $K_{\mathrm{PH}}^{\mathrm{mis}}$, $K_{\mathrm{PT}}^{\mathrm{mis}}$ and $K_{\mathrm{TH}}^{\mathrm{mis}}$ made to account for misalignment have estimates of unity. The alignment for each configuration is undertaken independently, and so the correlation associated with each pair of these corrections is expected to be weak, and is assumed to be zero.

3.4 Evaluate transfer impedances

The quantities Z_{PH}, Z_{PT} and Z_{TH} representing the transfer impedances in the three configurations of the devices are given by

$$Z_{\mathrm{PH}} = \frac{V_{\mathrm{PH}}}{I_{\mathrm{PH}}}, \quad Z_{\mathrm{PT}} = \frac{V_{\mathrm{PT}}}{I_{\mathrm{PT}}}, \quad Z_{\mathrm{TH}} = \frac{V_{\mathrm{TH}}}{I_{\mathrm{TH}}}. \tag{15}$$

Using expressions (10) to (15),

$$Z_{\mathrm{PH}} = \widehat{Z}_{\mathrm{PH}} \delta Z_{\mathrm{PH}}, \quad Z_{\mathrm{PT}} = \widehat{Z}_{\mathrm{PT}} \delta Z_{\mathrm{PT}}, \quad Z_{\mathrm{TH}} = \widehat{Z}_{\mathrm{TH}} \delta Z_{\mathrm{TH}}, \tag{16}$$

where

$$\widehat{Z}_{\mathrm{PH}} = \frac{\widehat{V}_{\mathrm{PH}}}{\widehat{E}_{\mathrm{PH}} \widehat{C}^{\mathrm{F}} \widehat{A}_{\mathrm{PH}}}, \quad \widehat{Z}_{\mathrm{PT}} = \frac{\widehat{V}_{\mathrm{PT}}}{\widehat{E}_{\mathrm{PT}} \widehat{C}^{\mathrm{F}} \widehat{A}_{\mathrm{PT}}}, \quad \widehat{Z}_{\mathrm{TH}} = \frac{\widehat{V}_{\mathrm{TH}}}{\widehat{E}_{\mathrm{TH}} \widehat{C}^{\mathrm{F}} \widehat{A}_{\mathrm{TH}}}, \tag{17}$$

are the calculated values of the transfer impedances, and

$$\delta Z_{\mathrm{PH}} = \frac{\delta V_{\mathrm{PH}} K_{\mathrm{PH}}^{\mathrm{sp}} K_{\mathrm{PH}}^{\mathrm{ss}} K_{\mathrm{PH}}^{\mathrm{load}} K_{\mathrm{PH}}^{\mathrm{mis}}}{\delta E_{\mathrm{PH}} C^{\mathrm{cal}} C^{\mathrm{cor}} C_{\mathrm{PH}}^{\mathrm{lin}} \delta A_{\mathrm{PH}}}, \tag{18}$$

with similar expressions for δZ_{PT} and δZ_{TH}, are quantities representing the relative deviations of the transfer impedances from their calculated values and are expressed in terms of quantities all of which have estimates of unity.

3.5 Evaluate hydrophone sensitivity

The quantity M_{H} representing the free-field sensitivity of the hydrophone H is given by (cf. formula (8))

$$M_{\mathrm{H}} = \delta M_{\mathrm{H}}^{\mathrm{rep}} \sqrt{K^{\mathrm{rec}} \frac{2}{\rho f} \frac{d_{\mathrm{PH}} d_{\mathrm{TH}}}{d_{\mathrm{PT}}} \frac{Z_{\mathrm{PH}} Z_{\mathrm{TH}}}{Z_{\mathrm{PT}}}}, \tag{19}$$

where

1. $\delta M_{\mathrm{H}}^{\mathrm{rep}}$ is a quantity describing the repeatability of the measurement of hydrophone sensitivity,
2. K^{rec} is a quantity describing a correction due to the non-reciprocal behaviour of the transducer T,
3. $\rho = \widehat{\rho} \delta \rho$ is the water density with measured value $\widehat{\rho}$,
4. $f = \widehat{f} \delta f$ is the acoustic frequency with measured value $\widehat{f}$, and
5. $d_{\mathrm{PH}} = \widehat{d}_{\mathrm{PH}} \delta d_{\mathrm{PH}}$, $d_{\mathrm{PT}} = \widehat{d}_{\mathrm{PT}} \delta d_{\mathrm{PT}}$ and $d_{\mathrm{TH}} = \widehat{d}_{\mathrm{TH}} \delta d_{\mathrm{TH}}$ are the distances between the devices in the three configurations with, respectively, measured values $\widehat{d}_{\mathrm{PH}}$, $\widehat{d}_{\mathrm{PT}}$ and $\widehat{d}_{\mathrm{TH}}$ obtained independently.

The estimate of $M_{\rm H}$ is obtained as the average of n values obtained independently by repeating the complete measurement. The corresponding random error quantity $\delta M_{\rm H}^{\rm rep}$ has an estimate of unity, a relative standard deviation assigned on the basis of a history of measured values, and is characterized by a Gaussian distribution.

The calibration method assumes that the transducer T is reciprocal, i.e., the free-field sensitivity and transmitting response of the device are related as in expression (5). If P is also assumed to be a reciprocal device, a fourth measurement can be made in which T is used as the transmitting device and P as the receiving device. Then, if T and P are both reciprocal, the measured values of the transfer impedances $Z_{\rm PT}$ and $Z_{\rm TP}$ should be consistent (accounting for measurement uncertainty). The difference between the values is used as the basis for associating an uncertainty with the estimate of unity for the quantity $K^{\rm rec}$ used to describe a correction for non-reciprocal behaviour. Specifically, the quantity is characterized by a rectangular distribution with a semi-width equal to half the difference between the measured values.[1]

The deviations $\delta d_{\rm PH}$, $\delta d_{\rm PT}$ and $\delta d_{\rm TH}$ of the distances between the devices from their measured values account for imperfections in the motion of the carriages of the system used to position the devices within the tank as well as other effects, such as deviations from vertical or bending of the poles (which are $3.5\,\rm m$ long) to which the devices are attached. The deviations have estimates of unity, and are characterized by rectangular distributions.

Using expressions (16) and (19),

$$M_{\rm H} = \widehat{M}_{\rm H}\delta M_{\rm H}, \qquad (20)$$

where

$$\widehat{M}_{\rm H} = \sqrt{\frac{2}{\widehat{\rho}\widehat{f}}\frac{\widehat{d}_{\rm PH}\widehat{d}_{\rm TH}}{\widehat{d}_{\rm PT}}\frac{\widehat{Z}_{\rm PH}\widehat{Z}_{\rm TH}}{\widehat{Z}_{\rm PT}}} \qquad (21)$$

is the calculated value of $M_{\rm H}$, and

$$\delta M_{\rm H} = \delta M_{\rm H}^{\rm rep}\sqrt{K^{\rm rec}\frac{1}{\delta\rho\delta f}\frac{\delta d_{\rm PH}\delta d_{\rm TH}}{\delta d_{\rm PT}}\frac{\delta Z_{\rm PH}\delta Z_{\rm TH}}{\delta Z_{\rm PT}}} \qquad (22)$$

is a quantity representing the relative deviation of $M_{\rm H}$ from its calculated value.

[1]The approximation to the semi-width of the rectangular distribution used to characterize $K^{\rm rec}$ calculated on the basis of the two measured values can be a poor one. Consideration is given in section 5 to the impact on the results of regarding the approximation as unreliable (for this and the other quantities in the measurement model characterized by rectangular distributions), and it is shown that the impact is small. A treatment of the problem of determining from a small number of measured values the semi-width of a rectangular distribution used to characterize a quantity is available [7].

3.6 Formulation

The output quantity Y is the relative deviation δM_{H} of the hydrophone sensitivity M_{H} from its calculated value $\widehat{M}_{\mathrm{H}}$. The input quantities $\boldsymbol{X}$ are listed in table 1, together with the uncertainties associated with the estimates (all of which are unity) of the quantities and the probability distributions used to characterize them. In the case of a quantity characterized by a normal or Gaussian distribution, the uncertainty is reported as a relative standard deviation (as a percentage). Where a quantity is characterized by a rectangular distribution, the uncertainty is reported as the semi-width of the distribution (also as a percentage).

The mathematical relationship between the output quantity Y and input quantities $\boldsymbol{X}$ takes the form of the measurement function $Y = f(\boldsymbol{X})$ defined by expression (22) with δZ_{PH} given by expression (18) with similar expressions for δZ_{PT} and δZ_{TH}. If correlations associated with pairs of input quantities (such as corrections) in the model are treated, this is done by expressing the quantities in terms of (other) independent quantities as described in section 3.7.

The quantity G in the sub-models of sections 3.1 and 3.3 is not included as an input quantity because the quantity cancels in the calculation of each transfer impedance (cf. expression (18)).

3.7 Treatment of correlated effects

Consider the quantities K_{PH}, K_{PT} and K_{TH} describing corrections in the three configurations of the devices for (a) lack of spherical spreading or (b) lack of steady state or (c) the effects of electrical loading (section 3.3). Express these quantities in terms of independent component quantities as follows:

$$
\begin{aligned}
K_{\mathrm{PH}} &= (1 + K_{\mathrm{PH,PH}})(1 + K_{\mathrm{PH,PT}})(1 + K_{\mathrm{PH,TH}}),\\
K_{\mathrm{PT}} &= (1 + K_{\mathrm{PT,PT}})(1 + K_{\mathrm{PT,PH}})(1 + K_{\mathrm{PT,TH}}),\\
K_{\mathrm{TH}} &= (1 + K_{\mathrm{TH,TH}})(1 + K_{\mathrm{TH,PH}})(1 + K_{\mathrm{TH,PT}}),
\end{aligned}
$$

in which $K_{\mathrm{A,B}}$ denotes a correction for configuration A that depends on that for configuration B (with A and B each denoting PH, PT or TH). The correction $K_{\mathrm{A,B}}$ has an estimate of zero for all configurations A and B. To a first order approximation, the above expressions take the form

$$
\begin{aligned}
K_{\mathrm{PH}} &= 1 + K_{\mathrm{PH,PH}} + K_{\mathrm{PH,PT}} + K_{\mathrm{PH,TH}},\\
K_{\mathrm{PT}} &= 1 + K_{\mathrm{PT,PT}} + K_{\mathrm{PT,PH}} + K_{\mathrm{PT,TH}},\\
K_{\mathrm{TH}} &= 1 + K_{\mathrm{TH,TH}} + K_{\mathrm{TH,PH}} + K_{\mathrm{TH,PT}},
\end{aligned}
$$

so that the correction K_{A} has an estimate of unity for all configurations A.

Let $u(K_{\mathrm{A}})$ denote the standard deviation of K_{A} (and similarly for the other quantities) and $u(K_{\mathrm{A}}, K_{\mathrm{B}}) = r_{\mathrm{A,B}}u(K_{\mathrm{A}})u(K_{\mathrm{B}})$ the covariance of K_{A} and K_{B} (and similarly for the other

Input quantity	Distribution	Uncertainty/%
δE_{PH}, δE_{PT}, δE_{TH}	N	0.50
C^{cal}	N	0.75
C^{cor}	R	0.50
$C^{\mathrm{lin}}_{\mathrm{PH}}$, $C^{\mathrm{lin}}_{\mathrm{PT}}$, $C^{\mathrm{lin}}_{\mathrm{TH}}$	R	1.0
δA_{PH}, δA_{PT}, δA_{TH}	N	0.10
δV_{PH}, δV_{PT}, δV_{TH}	N	0.50
$K^{\mathrm{sp}}_{\mathrm{PH}}$, $K^{\mathrm{sp}}_{\mathrm{PT}}$, $K^{\mathrm{sp}}_{\mathrm{TH}}$	R	2.0
$K^{\mathrm{ss}}_{\mathrm{PH}}$, $K^{\mathrm{ss}}_{\mathrm{PT}}$, $K^{\mathrm{ss}}_{\mathrm{TH}}$	R	2.0
$K^{\mathrm{load}}_{\mathrm{PH}}$, $K^{\mathrm{load}}_{\mathrm{PT}}$, $K^{\mathrm{load}}_{\mathrm{TH}}$	R	1.0
$K^{\mathrm{mis}}_{\mathrm{PH}}$, $K^{\mathrm{mis}}_{\mathrm{PT}}$, $K^{\mathrm{mis}}_{\mathrm{TH}}$	R	1.0
K^{rec}	R	1.5
$\delta M^{\mathrm{rep}}_{\mathrm{H}}$	N	1.5
$\delta\rho$	R	0.20
δf	R	0.20
δd_{PH}, δd_{PT}, δd_{TH}	R	1.0

Table 1: Input quantities and the probability distributions used to characterize them. For a normal distribution (N) the relative standard deviation of the quantity is given (as a percentage), and for a rectangular distribution (R) the semi-width is given (also as a percentage).

pairs of quantities), where $r_{\mathrm{A,B}}$ is the correlation coefficient associated with K_{A} and K_{B}. Assume $u(K_{\mathrm{A}}) = u(K)$ for all configurations A.

Two cases are considered.

Firstly, suppose $K_{\mathrm{A,B}} = K_{\mathrm{B,A}}$ for all configurations A and B, i.e., each pair configurations A and B depends on a pair of quantities that are identical in magnitude and sign so that the quantities are *positively* correlated. Then, to a first order approximation,

$$u^2(K_{\mathrm{A}}) = u^2(K) = u^2(K_{\mathrm{A,A}}) + u^2(K_{\mathrm{A,B}}) + u^2(K_{\mathrm{A,C}})$$

and

$$u(K_{\mathrm{A}}, K_{\mathrm{B}}) = r_{\mathrm{A,B}} u^2(K) = u^2(K_{\mathrm{A,B}}).$$

It follows that

$$u^2(K_{\mathrm{A,B}}) = r_{\mathrm{A,B}} u^2(K), \quad u^2(K_{\mathrm{A,A}}) = (1 - r_{\mathrm{A,B}} - r_{\mathrm{A,C}})\, u^2(K), \tag{23}$$

which expresses the standard deviations of the independent component quantities in terms of $u^2(K)$ and correlation coefficients $r_{\mathrm{A,B}}$ satisfying

$$0 \le r_{\mathrm{A,B}} \le 1, \quad 0 \le r_{\mathrm{A,B}} + r_{\mathrm{A,C}} \le 1.$$

Secondly, suppose $K_{\mathrm{A,B}} = -K_{\mathrm{B,A}}$ for all configurations A and B, i.e., each pair configurations A and B depends on a pair of quantities that are identical in magnitude and but have opposite sign so that the quantities are *negatively* correlated. Then, to a first order approximation,

$$u^2(K_{\mathrm{A}}) = u^2(K) = u^2(K_{\mathrm{A,A}}) + u^2(K_{\mathrm{A,B}}) + u^2(K_{\mathrm{A,C}})$$

and

$$u(K_{\mathrm{A}}, K_{\mathrm{B}}) = r_{\mathrm{A,B}} u^2(K) = -u^2(K_{\mathrm{A,B}}).$$

It follows that

$$u^2(K_{\mathrm{A,B}}) = -r_{\mathrm{A,B}} u^2(K), \quad u^2(K_{\mathrm{A,A}}) = (1 + r_{\mathrm{A,B}} + r_{\mathrm{A,C}})\, u^2(K), \tag{24}$$

which expresses the standard deviations of the independent component quantities in terms of $u^2(K)$ and correlation coefficients $r_{\mathrm{A,B}}$ satisfying

$$-1 \le r_{\mathrm{A,B}} \le 0, \quad -1 \le r_{\mathrm{A,B}} + r_{\mathrm{A,C}} \le 0.$$

Assuming the correlation coefficients all have the same sign, expressions (23) and (24) can be stated as

$$u^2(K_{\mathrm{A,B}}) = |r_{\mathrm{A,B}}| u^2(K), \quad u^2(K_{\mathrm{A,A}}) = (1 - |r_{\mathrm{A,B}}| - |r_{\mathrm{A,C}}|)\, u^2(K), \tag{25}$$

with

$$|r_{\mathrm{A,B}}| + |r_{\mathrm{A,C}}| \le 1. \tag{26}$$

Expression (25) provides a means to evaluate the standard deviations of the independent component quantities in terms of prescribed values for the (combined) standard deviation $u(K)$ of the quantities $K_{\rm A}$ and the correlation coefficients (all positive or all negative) associated with pairs of those quantities. Condition (26) says that the correlation coefficients associated with pairs of the quantities cannot be set independently. In practice, the correlation coefficients are not known or are difficult to quantify, and an uncertainty analysis is undertaken for different values of the coefficients in order to investigate the sensitivity of the results to those values.

The analysis presented above shows how to decompose the (combined) standard deviation $u(K)$ of the quantities $K_{\rm A}$ into standard deviations for the independent component quantities in order to achieve prescribed correlation coefficients for pairs of the quantities $K_{\rm A}$. The decomposition is approximate because it depends on a first order approximation for the measurement function used to express $K_{\rm A}$ in terms of those component quantities. Furthermore, the decomposition gives no information about how to choose the probability distributions for the component quantities to achieve a prescribed distribution for $K_{\rm A}$. If the component quantities are characterized by Gaussian distributions, the distribution for $K_{\rm A}$ can be expected to be approximately Gaussian. However, unless one of the component quantities is dominant and is characterized by a rectangular distribution, the distribution for $K_{\rm A}$ cannot be expected to be rectangular.

For the corrections for lack of spherical spreading (or those for the effects of electrical loading), it is assumed that $r^{\rm sp}_{\rm A,B} = r$ (or $r^{\rm load}_{\rm A,B} = r$) for all pairs of configurations A and B. Then, expression (25) and condition (26) become

$$u^2(K_{\rm A,B}) = |r| u^2(K), \quad u^2(K_{\rm A,A}) = (1 - 2|r|)\, u^2(K),$$

for all configurations, with

$$|r| \le 1/2.$$

On the other hand, for the corrections for lack of spherical spreading, it is assumed that $r^{\rm ss}_{\rm PH,PT} = r^{\rm ss}$ is the only non-zero correlation coefficient. Then, expression (25) and condition (26) become

$$u^2(K_{\rm PH,PT}) = |r^{\rm ss}| u^2(K), \quad u^2(K_{\rm PH,PH}) = u^2(K_{\rm PT,PT}) = (1 - |r^{\rm ss}|)\, u^2(K)$$

and

$$u^2(K_{\rm TH,TH}) = u^2(K),$$

with the standard deviation for all other component quantities equal to zero, and

$$|r^{\rm ss}| \le 1.$$

4 Measurement uncertainty evaluation

4.1 The propagation of probability distributions

The basis for the evaluation of measurement uncertainty is the propagation of probability distributions. In order to apply the propagation of probability distributions, a measurement function of the generic form $Y = f(X_1, \ldots, X_N)$ relating input quantities $X_1, \ldots, X_N$, about which information is available, and the output quantity Y, about which information is required, is formulated. Additionally, information concerning the input quantities is encoded as probability distributions for those quantities, such as rectangular (uniform), Gaussian (normal), etc. The information can take a variety of forms, including a series of indication values, data on a calibration certificate, and the expert knowledge of the metrologist. An implementation of the propagation of probability distributions provides a probability distribution for Y, from which can be obtained an estimate of Y, the standard uncertainty associated with the estimate, and a coverage interval for Y corresponding to a stipulated (coverage) probability. Particular implementations of the approach are the GUM uncertainty framework [1, 6] (section 4.2) and a Monte Carlo method [3, 4, 6] (section 4.3).

4.2 GUM uncertainty framework

The primary guide in metrology on uncertainty evaluation is the 'Guide to the expression of uncertainty in measurement' (GUM) [1, 2]. It presents a framework for uncertainty evaluation based on the use of the law of propagation of uncertainty and the central limit theorem. The law of propagation of uncertainty provides a means for 'propagating uncertainties' through the measurement function, i.e., for evaluating the standard uncertainty $u(y)$ associated with the estimate $y = f(x_1, \ldots, x_N)$ of Y given the standard uncertainties $u(x_i)$ associated with the estimates x_i of X_i (and, when they are non-zero, the covariances $u(x_i, x_j)$ associated with pairs of estimates x_i and x_j). The central limit theorem is applied to characterize Y by a Gaussian distribution (or, in the case of finite effective degrees of freedom, by a t-distribution), which is used as the basis of providing a coverage interval for Y.

For the case of input quantities that are independent, the law of propagation of uncertainty takes the general form

$$u^2(y) = c_1^2 u^2(x_1) + c_2^2 u^2(x_2) + \cdots c_N^2 u^2(x_N),$$

where c_i is the sensitivity coefficient for X_i given by the first order partial derivative of f with respect to X_i evaluated at the estimates of the input quantities. For the measurement function of concern here, the sensitivity coefficients are straightforward to calculate and the law of propagation of uncertainty takes a particularly simple form. Consider the measurement function

$$Y = X_1 \sqrt{\frac{X_2}{X_3}}$$

of $N = 3$ input quantities, which is indicative of the measurement function (22), with estimates $x_1 = x_2 = x_3 = 1$. Then,

$$c_1 = \sqrt{\frac{x_2}{x_3}} = 1, \quad c_2 = \frac{x_1}{2}\sqrt{\frac{1}{x_2 x_3}} = \frac{1}{2}, \quad c_3 = -\frac{x_1}{2}\sqrt{\frac{x_2}{x_3^3}} = -\frac{1}{2},$$

and hence

$$u^2(y) = u^2(x_1) + \left(\frac{1}{2}\right)^2 u^2(x_2) + \left(\frac{1}{2}\right)^2 u^2(x_3).$$

4.3 Monte Carlo method

A Monte Carlo method for uncertainty evaluation is based on the following consideration [4, 5]. The estimate y of Y is conventionally obtained, as in section 4.2, by evaluating the measurement function for the estimates x_i of X_i. However, since each X_i is described by a probability distribution, a value as legitimate as x_i can be obtained by drawing a value at random from the distribution. The method operates, therefore, in the following manner. A random draw is made from the probability distribution for each X_i and the corresponding value of Y is formed by evaluating the measurement function for these values. Many Monte Carlo trials are performed, i.e., the process is repeated many times, to obtain M, say, values $y_k,\ k = 1, \ldots, M$, of Y. The values y_k are used to provide an approximation to the probability distribution for Y, and thence an estimate of Y, the standard uncertainty associated with the estimate and a coverage interval for Y.

4.4 Comparison of methods

Although the GUM uncertainty framework can be expected to work well in many circumstances, it is generally difficult to quantify the effects of the approximations involved, which include the linearization of the measurement function in the application of the law of propagation of uncertainty, the evaluation of effective degrees of freedom using the Welch-Satterthwaite formula, and the assumption that the output quantity can be characterized by a Gaussian distribution (or t-distribution). In contrast, the Monte Carlo method has a number of features [5], including (a) that it is applicable regardless of the nature of the measurement function, i.e., whether it is linear, mildly non-linear or highly non-linear, (b) that there is no requirement to evaluate effective degrees of freedom, and (c) that no assumption is made about the distribution for Y, for example, that it is Gaussian. In consequence, the Monte Carlo method provides results that are free of the approximations involved in applying the GUM uncertainty framework, and it can be expected to provide an uncertainty evaluation that is reliable for a wide range of measurement problems. Furthermore, the method can be used, as here, as a basis for validating the results obtained using the GUM uncertainty framework and testing the assumptions made in applying the GUM uncertainty framework as an approach to measurement uncertainty evaluation.

An important distinction between the two approaches to measurement uncertainty evaluation is that the GUM uncertainty framework only makes use of expectations and standard deviations (and possibly covariances and degrees of freedom) that are used as summaries of the probability distributions used to characterize the input quantities. In contrast, the Monte Carlo method uses the complete information about the input quantities encoded by these distributions.

5 Results

The results presented in this section are indicative of a calibration undertaken at the frequency $f = 50\,\mathrm{kHz}$.[2]

The GUM uncertainty framework returns the estimate $y = 1.000\,0$ with associated relative standard uncertainty $u(y) = 2.45\,\%$. The Gaussian probability density function that characterizes the output quantity on the basis of this information is shown as the solid (red) curve in figure 2. The endpoints of a $95\,\%$ coverage interval for the output quantity are shown as dashed (red) vertical lines. An application of a Monte Carlo method with $M = 10^7$ trials returns the estimate $y = 1.000\,1$ with associated relative standard uncertainty $u(y) = 2.45\,\%$. An approximation to the probability density function that characterizes the output quantity provided by a Monte Carlo method is shown as the histogram (scaled frequency distribution) in figure 2. The endpoints of a $95\,\%$ coverage interval for the output quantity are shown as solid (blue) vertical lines. There is very good agreement between the results provided by the two approaches to uncertainty evaluation.

Table 2 lists the input quantities X_i (as in table 1), the relative standard uncertainties associated with estimates $x_i = 1$ of the X_i, and the contributions $u_i(y)$ of each input quantity to the (combined) relative standard uncertainty $u(y)$ associated with an estimate y of the output quantity Y provided by the GUM uncertainty framework (third column) and a Monte Carlo method (fourth to sixth columns). In rows of the table in which three input quantities are listed, the contributions evaluated using the GUM uncertainty framwork are equal (and the single value is given) whereas they can be different when evaluated using a Monte Carlo method. Again, there is good agreement between the results provided by the two approaches to uncertainty evaluation. The dominant contributions to $u(y)$ come from $\delta M_{\mathrm{H}}^{\mathrm{rep}}$, the quantity describing the repeatability of the measurement of hydrophone sensitivity, and the corrections $K_{\mathrm{AB}}^{\mathrm{sp}}$ for lack of spherical spreading and $K_{\mathrm{AB}}^{\mathrm{ss}}$ for lack of steady state in the different configurations. Together, these seven input quantities (from the total of 33) account for almost $85\,\%$ of the (combined) relative standard uncertainty associated with the estimate of the output quantity.

Figure 3 shows the impact on the results provided by the GUM uncertainty framework of

[2]Results are given to a number of significant decimal digits to expose differences between the results obtained using the GUM uncertainty framework and a Monte Carlo method. In practice, uncertainties would be reported to one or two significant decimal digits.

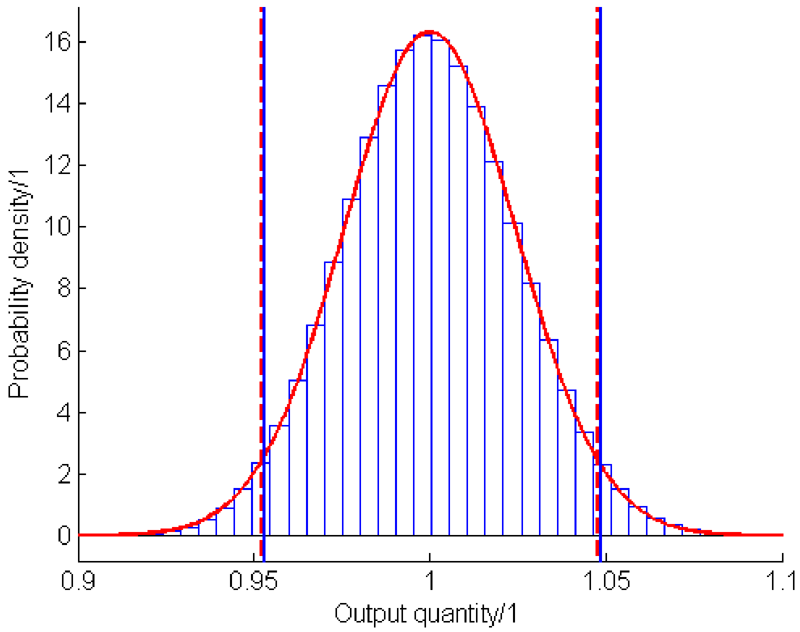

Figure 2: Probability density functions for δM_{H} obtained using the GUM uncertainty framework (solid curve) and a Monte Carlo method (histogram). The endpoints of the $95\,\%$ coverage intervals provided by the two approaches are also indicated.

different choices for the values of the correlation coefficients associated with pairs of estimates of the corrections for (a) lack of spherical spreading, (b) lack of steady state, and (c) electrical loading, keeping the total relative standard deviations the same for each correction. In all three cases, the impact of a positive correlation is to reduce the (combined) relative standard uncertainty $u(y)$, with the reduction greatest for correlation associated with pairs of estimates of the corrections for lack of steady state. Consequently, treating the quantities as independent provides a 'safe' value for $u(y)$ in the case that pairs of quantities are positively correlated but it is difficult to quantify the correlation coefficients. Positive correlation associated with pairs of quantities leads to the 'cancellation' of quantities in the measurement function (cf. the cancellation of the quantity G in the sub-models of sections 3.1 and 3.3 in the calculation of each transfer impedance (cf. expression (18)). On the other hand, negative correlation will, in this example, tend to 'reinforce' quantities, leading to an increase in $u(y)$.

The GUM uses (finite) degrees of freedom to describe imprecise knowledge about the uncertainty associated with an estimate of an input quantity. When the uncertainty is assigned on the basis of repeated indication values obtained independently, the degrees of freedom ν is evaluated in terms of the number n of values using $\nu = n-1$. When the uncertainty is assigned otherwise (as for a quantity characterized by a rectangular distribution), the degrees of freedom is used to express the 'reliability' (relative uncertainty) of the standard deviation of the quantity. For example, if the knowledge of how an estimate x_i was determined and the associated standard uncertainty $u(x_i)$ evaluated leads to the judgement that the value of $u(x_i)$ is reliable to about $25\,\%$, then the degrees of freedom ν associated with $u(x_i)$ is $\nu = 8$ [1, clause G.4.2, example].

Imprecise knowledge can also be encoded in the distribution used to characterize the quantity [3, clause 6]. A t-distribution with $\nu = n - 1$ degrees of freedom is used (in place of a Gaussian distribution) when the information about a quantity takes the form of n repeated indication values. When a quantity is known to lie between limits A and B, and the midpoint $(A + B)/2$ of the interval defined by the limits is fixed but the length $B - A$ is not known exactly, the quantity is characterized by a curvilinear trapezoidal distribution (in place of a rectangular distribution). For example, it may be known that A lies in the interval $a \pm d$ and B in $b \pm d$ where a, b and d are specified.

As an example, suppose for each input quantity listed in table 1 that is characterized by a rectangular distribution, a curvilinear trapezoidal distribution is used instead, with the parameter d of the distribution set as $d = 0.5 \times (b-a)/2$, i.e., the semi-width of the distribution is set with a 'reliability' of $50\,\%$. In this case, the GUM uncertainty framework returns the estimate $y = 1.000\,0$ and associated relative standard uncertainty $u(y) = 2.45\,\%$, which is the same as before because the evaluation of y and $u(y)$ does not depend on the values of the degrees of freedom. Application of the Welch-Satterthwaite formula gives an effective degrees of freedom $\nu_{\text{eff}} = 91$, so there is a very modest increase in the coverage factor for the calculation of a coverage interval by this approach. An application of a Monte Carlo method with $M = 10^7$ trials returns the estimate $y = 1.000\,1$ with associated relative standard uncertainty $u(y) = 2.50\,\%$. The effect on the results of accounting for imprecise knowledge about these quantities is quite modest.

Input quantity X_i	$u(x_i)$/%	$u_i(y)$/%			
		GUF	MCM		
$\delta E_{\mathrm{PH}}, \delta E_{\mathrm{PT}}, \delta E_{\mathrm{TH}}$	0.500 0	0.250 0	0.250 0	0.249 9	0.250 0
C^{cal}	0.750 0	0.375 0	0.375 2		
C^{cor}	0.288 7	0.144 3	0.144 3		
$C^{\mathrm{lin}}_{\mathrm{PH}}, C^{\mathrm{lin}}_{\mathrm{PT}}, C^{\mathrm{lin}}_{\mathrm{TH}}$	0.577 4	0.288 7	0.288 7	0.288 8	0.288 7
$\delta A_{\mathrm{PH}}, \delta A_{\mathrm{PT}}, \delta A_{\mathrm{TH}}$	0.100 0	0.050 0	0.050 0	0.050 0	0.050 0
$\delta V_{\mathrm{PH}}, \delta V_{\mathrm{PT}}, \delta V_{\mathrm{TH}}$	0.500 0	0.250 0	0.250 0	0.250 1	0.250 0
$K^{\mathrm{sp}}_{\mathrm{PH}}, K^{\mathrm{sp}}_{\mathrm{PT}}, K^{\mathrm{sp}}_{\mathrm{TH}}$	1.154 7	0.577 4	0.577 5	0.577 6	0.577 5
$K^{\mathrm{ss}}_{\mathrm{PH}}, K^{\mathrm{ss}}_{\mathrm{PT}}, K^{\mathrm{ss}}_{\mathrm{TH}}$	1.154 7	0.577 4	0.577 3	0.577 4	0.577 3
$K^{\mathrm{load}}_{\mathrm{PH}}, K^{\mathrm{load}}_{\mathrm{PT}}, K^{\mathrm{load}}_{\mathrm{TH}}$	0.577 4	0.288 7	0.288 6	0.288 7	0.288 7
$K^{\mathrm{mis}}_{\mathrm{PH}}, K^{\mathrm{mis}}_{\mathrm{PT}}, K^{\mathrm{mis}}_{\mathrm{TH}}$	0.577 4	0.288 7	0.288 7	0.288 7	0.288 7
K^{rec}	0.866 0	0.433 0	0.432 9		
$\delta M^{\mathrm{rep}}_{\mathrm{H}}$	1.500 0	1.500 0	1.500 1		
$\delta\rho$	0.115 5	0.057 7	0.057 7		
δf	0.115 5	0.057 7	0.057 8		
$\delta d_{\mathrm{PH}}, \delta d_{\mathrm{PT}}, \delta d_{\mathrm{TH}}$	0.577 4	0.288 7	0.288 6	0.288 6	0.288 7

Table 2: Input quantities X_i, relative standard uncertainties $u(x_i)$, and contributions $u_i(y)$ to $u(y)$ obtained using the GUM uncertainty framework (GUF) and a Monte Carlo method (MCM).

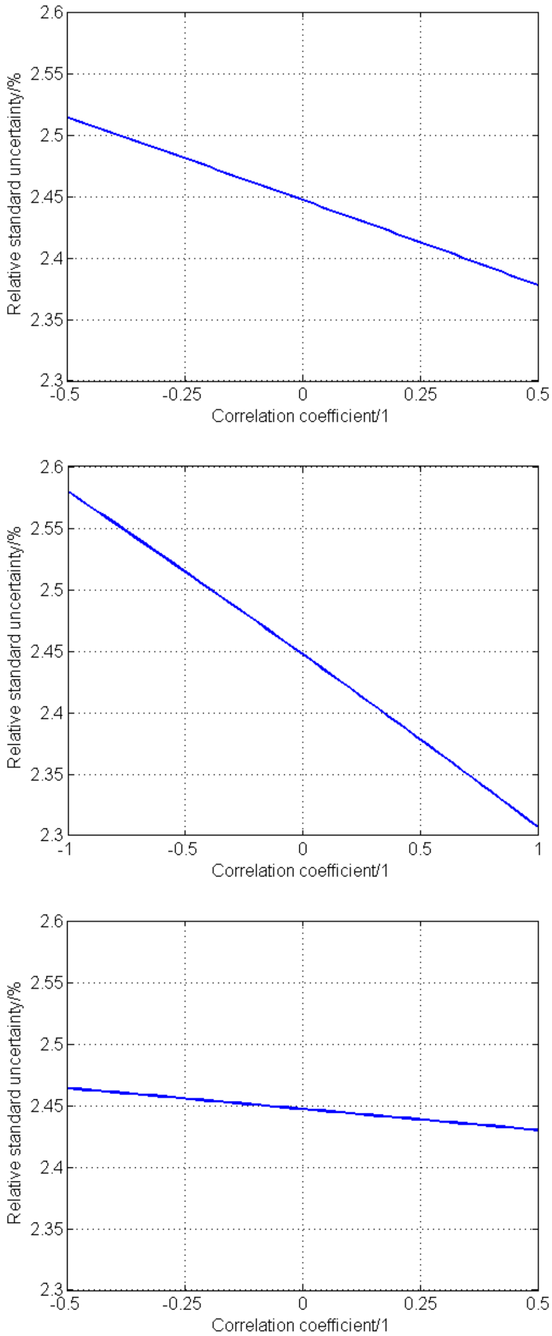

Figure 3: Standard uncertainty (as a percentage) as a function of the correlation coefficient $r^{\mathrm{sp}}_{\mathrm{A,B}}$ (top), $r^{\mathrm{ss}}_{\mathrm{A,B}}$ (middle) and $r^{\mathrm{load}}_{\mathrm{A,B}}$ (bottom).

6 Conclusions

This report has been concerned with the evaluation of measurement uncertainty associated with estimates of the transmitting and receiving sensitivities of an underwater acoustic transducer provided by NPL's implementation of the calibration method of three-transducer spherical-wave reciprocity. Consideration has been given to formulating the problem of uncertainty evaluation and comparing the results obtained using conventional 'uncertainty propagation' and a Monte Carlo method as approaches to solving the formulated problem. The sensitivity of the results obtained to assumptions about the correlations associated with influence quantities in the calibration and the reliability of the available information about those quantities has been investigated.

The conclusions of the work are summarized as follows:

1. The measurement function, which expresses the output quantity or measurand (transmitting or receiving sensitivity at a given frequency) in terms of input quantities involved in the measurement, is a reasonably complicated function. The function involves 33 input quantities some of which are mutually dependent. The measurement function (or the measurement model on which it is based) is developed from a sequence of sub-models that describe different aspects of the measurement, such as the measurement of electrical voltages and currents and the evaluation of transfer impedances.

2. The results obtained using the GUM uncertainty framework (or conventional 'uncertainty propagation') are validated by those obtained using a Monte Carlo method. Consequently, it is appropriate to use the GUM uncertainty framework as an approach to uncertainty evaluation for the problem described as well as for sufficiently similar problems.

3. The impact of positive correlation associated with quantities in the measurement function describing corrections for lack of spherical spreading, lack of steady state and electrical loading is to reduce the (combined) standard uncertainty. Consequently, treating the quantities as independent is a 'safe' assumption about the quantities.

4. The impact of accounting for imprecise information about quantities characterized by rectangular distributions (such as those for which the information is based on 'expert knowledge') is quite modest. Furthermore, the impact is quantifiable. For example, if the semi-widths of those distributions are regarded as 'reliable' to $50\,\%$, the standard uncertainty provided by a Monte Carlo method remains unchanged when reported to two significant decimal digits.

Acknowledgements

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