

Report

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**Magnitude and phase
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reference broadband
membrane hydrophone up to
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OCTOBER 2007

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DQL

ABSTRACT

A feasibility study aimed at assessing and improving a hydrophone calibration method employing a combination of ultrasonic field measurement and nonlinear field prediction has been carried out. The magnitude and relative phase characteristics of a broadband membrane hydrophone were obtained between 5 MHz and 100 MHz by comparing its open-circuit output voltage with the ultrasonic pressure at the hydrophone location in a harmonically rich field, theoretically predicted using the KZK equation. The methodology employed differs from previous work (Cooling and Humphrey 2007, Bleeker and Lewin 2000) in that prior knowledge of the hydrophone sensitivity between 5 MHz and 40 MHz is assumed and that the predicted field is optimised against the measured field over this frequency range by varying the source and medium parameters. The predicted hydrophone magnitude response shows consistency within $\pm 10\%$ at transducer-hydrophone separations between 90 mm and 180 mm.

The radial distribution of the ultrasonic field has been mapped over the radius of the hydrophone active element at selected axial positions, hence providing further insight into effects of spatial averaging due to finite hydrophone element receiving area.

3.5 MHz fundamental frequency checksource waveforms were acquired using both the broadband membrane hydrophone characterised in this study and a laser interferometer. At an axial position where the frequency content of the acoustic pressure is within the bandwidth of the interferometer (51.5 mm), agreement for p_+ and p_- is within 3% and better than 1% for I_{spta} and t_d . At transducer-hydrophone separations where the frequency content of the acoustic pressure extends beyond the bandwidth of the interferometer (80 mm), p_+ appears underestimated by 14% in the interferometer measurement. This highlights the need for better measurement capability of laser interferometry beyond 50 MHz.

This method shows promise in assisting the development of laser interferometry beyond 50 MHz, aiding mapping of transducers fields in the axial and radial directions and estimating the magnitude and phase characteristics of a test hydrophone by using the characterized hydrophone as the reference device.

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Approved on behalf of the Managing Director, NPL
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1. INTRODUCTION

Currently, 25% of hydrophone calibrations provided to industry by NPL are carried out at frequencies above 20 MHz. In harmonic-rich fields commonly encountered in medical ultrasound applications, where as much as 40% of the energy is transferred from the fundamental frequency to higher order harmonics (Humphrey et al 2006), it is common for frequency content in excess of 80 MHz to occur. As such, there is a requirement for hydrophones to be calibrated up to frequencies as high as 100 MHz. In ultrasonic fields spanning a large frequency range, the phase response of the measurement system may be of crucial importance. If the phase response of the measurement system is not linear, each harmonic in the ultrasonic field will be affected by a different group delay as the hydrophone converts acoustic pressure into voltage via the transduction process. Temporal smearing of the ultrasonic pressure waveform may then occur, and relating the hydrophone output voltage waveform to the pressure waveform via the sensitivity of the measurement system at the fundamental frequency, as recommended in IEC 61157 (1992) may lead to uncertainties in the acoustic waveform parameters. For example, a 25 μm bilaminar membrane hydrophone has a sensitivity gradually increasing with frequency to a peak occurring at approximately 20 MHz, which arises due to thickness resonance. This increase in response strongly affects the measured waveform (Smith 1986). For a typical 3.5 MHz nonlinearly distorted waveform, it can lead to an error of $\pm 15\%$ to $\pm 19\%$ (Esward et al 2000) in the value for the temporal-average intensity, one of the important quantities required for compliance with regulatory standards.

In order to address this source of error and improve measurement accuracy, optical techniques have been successfully employed up to 70 MHz (Wilkins and Koch 2004). Currently, these methods are the most likely to enable primary standard calibration of hydrophones up to 100 MHz. Nevertheless, it is vital to develop other calibration methods to help provide confidence in optical methods.

Although simulating the transfer characteristics of membrane hydrophones has been successfully carried out in the past using one-dimensional modeling techniques (Gélat et al 2005), this does not constitute an attempt at calibrating the actual device.

As part of the work undertaken in assessing the ‘magnomic’ method (Zeqiri et al 1999), theoretical modeling of propagating acoustic plane waves of finite amplitude was used in conjunction with measurements from a 10 MHz planar transducer to calibrate a range of membrane hydrophones between 10 MHz and 60 MHz. Due to the underlying plane wave assumption, the ‘magnomic’ calibration method was deemed unsuitable.

A method that shows promise involves the comparison of a hydrophone-measured ultrasonic field with theoretical predictions resulting from field simulations. By making assumptions regarding the behaviour of the acoustic source, Bleeker and Lewin (2000) as well as Cooling and Humphrey (2006, 2007) have numerically solved the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation to characterise the ultrasonic pressure field. The hydrophone output voltage is obtained via measurement, and the hydrophone complex transfer characteristics may be obtained by comparison with the simulated pressure field.

In this report, a feasibility study aimed at assessing and improving the above methodology is carried out. Its objective is to obtain estimates of the magnitude and phase characteristics of a broadband membrane hydrophone up to 100 MHz.

2. METHOD

2.1 CHOICE OF TRANSDUCER AND FREQUENCY OF EXCITATION

As part of this feasibility study, it was required to characterise a reference broadband hydrophone up to a frequency of 100 MHz, by comparing its open-circuit output voltage with the theoretically predicted ultrasonic pressure at the hydrophone location. It was therefore necessary to generate signals with significant content at 100 MHz at the measurement positions considered. By exploiting effects of nonlinear propagation of finite amplitude ultrasonic waves in water, it is well known that integer multiples of the fundamental frequency can be generated in the propagating medium. As part of the work undertaken in assessing the ‘magnomic’ method (Zeqiri et al 1999), a 10 MHz planar transducer had been used to generate acoustic signals with frequency content beyond 80 MHz. It was therefore initially thought that a 10 MHz transducer could be considered for this investigation, albeit a focused device so as to maximise acoustic pressures at distances at which the reference hydrophone was to be characterised. A 5 MHz focused transducer was opted for as part of this work. This is particularly advantageous over a 10 MHz device, in that the frequency response of the hydrophone to be characterised will then be obtained in steps of 5 MHz rather than 10 MHz. When using this response for deconvolution purposes, less interpolation and extrapolation would be required to obtain data at intermediate frequencies and at frequencies between D.C. and 5 MHz.

In order to minimise the influence of spatial variation of the acoustic pressure field, the transducer should preferably be of moderate focal gain so that magnitude and phase of the acoustic pressure do not suffer from large variations at the measurement positions over the hydrophone receiving area.

For reasons to be detailed later, the theoretical model that was used to predict the ultrasonic pressure field will assume that the source behaves as an axisymmetric rigidly vibrating concave circular piston. Hence, uniform amplitude variation across the face of the transducer is to be assumed. Although it is well known that ultrasonic sources do not behave in this fashion (Guo et al 1992), (Hughes 2001), devices exist that can approximate this type of behaviour.

2.2 CHOICE OF BROADBAND HYDROPHONE MEASUREMENT SYSTEM

To carry out the measurements up to the required frequency as part of this feasibility study, a broadband hydrophone was required. A Marconi® 9 µm polyvinylidene fluoride (*pvd**f*) film thickness coplanar device with 0.5 mm active element diameter was used (S/N IP904). Although this device should in theory have a first resonance close to 90 MHz, previous measurements suggest that this could in fact be closer to 100 MHz (Bacon, 1982).

Other devices were considered for this study:

- Marconi® 9 µm *pvd**f* film thickness bilaminar device with 0.5 mm active element diameter (S/N IP016);

- Marconi® 9 μm *pvd*f film thickness coplanar device with 0.5 mm active element diameter (S/N IP901).

IP016 was deemed unsuitable due to its comparatively narrow bandwidth, its first resonance being around 60 MHz. Although some axial scans were acquired with IP901, changes in its sensitivity were subsequently noticed. It was therefore decided that this study would focus on characterising IP904.

In order to maximise signal-to-noise ratio at higher frequencies, IP904 was connected to an Onda® AH-2010-100 broadband amplifier. This amplifier features a flat voltage gain up to 100 MHz and a 72 dB dynamic range (Onda Corporation website). See Fig. 12.

Using this amplifier in conjunction with the reference broadband hydrophone created a measurement system with high gain and reasonably flat frequency response.

The equivalent electrical circuit of the configuration is shown in Fig. 1.

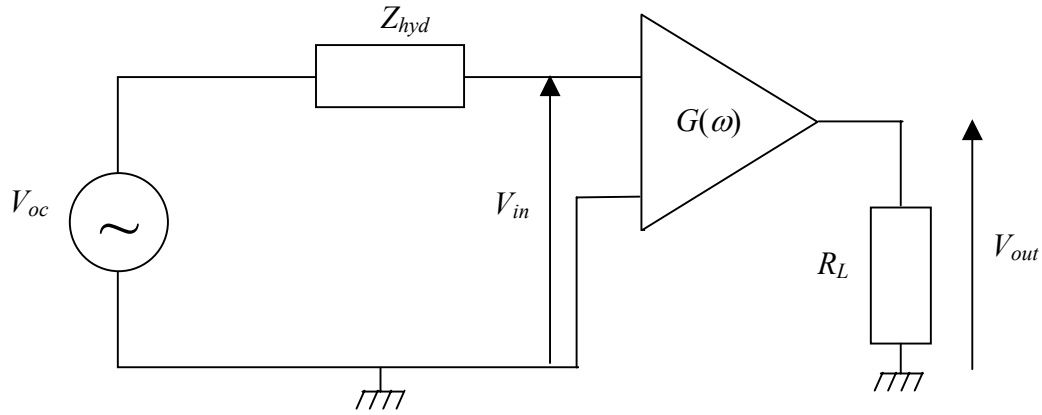


Fig. 1 – Hydrophone/amplifier equivalent electrical circuit.

- $V_{oc}(\omega)$ is the open circuit hydrophone output voltage for a pressure $P(\omega)$ at the hydrophone front face.
- $Z_{hyd}(\omega)$ is the complex hydrophone electrical impedance.
- $G(\omega)$ is the amplifier complex voltage gain.
- R_L is the load resistance equal to 50 Ω .

Let $Z_{in}(\omega)$ be the amplifier input impedance. The section of the electrical circuit in Fig. 1 at the amplifier input stage can be represented by the equivalent circuit in Fig 2.

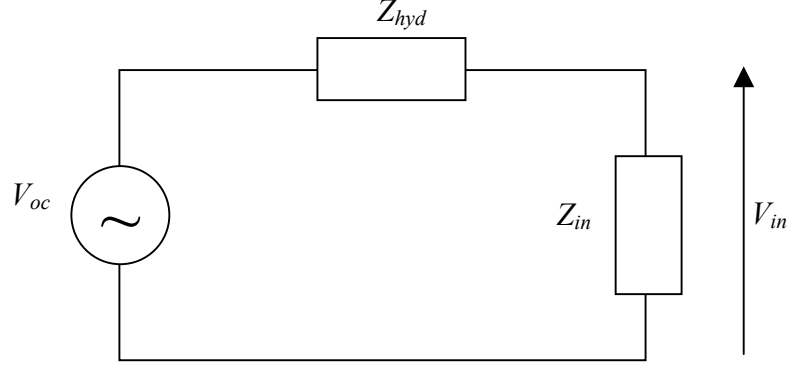


Fig 2 – Hydrophone/amplifier equivalent electrical circuit (amplifier input stage).

From Fig. 2, we have:

$$V_{in}(\omega) = \frac{Z_{in}(\omega)V_{oc}(\omega)}{Z_{in}(\omega) + Z_{hyd}(\omega)} \quad (1)$$

Since the amplifier has a voltage gain $G(\omega)$, the voltage at the amplifier output stage is given by:

$$V_{out}(\omega) = \frac{Z_{in}(\omega)V_{oc}(\omega)G(\omega)}{Z_{in}(\omega) + Z_{hyd}(\omega)} \quad (2)$$

Hence, the hydrophone open-circuit sensitivity is related to the loaded sensitivity via the following factor:

$$T(\omega) = \frac{Z_{in}(\omega)G(\omega)}{Z_{in}(\omega) + Z_{hyd}(\omega)} \quad (3)$$

The sensitivity of this hydrophone was characterised between 1-40 MHz using primary standard laser interferometry, as described by Esward and Robinson (1999).

2.3 SCANNING TANK SET-UP

The NPL Beam Plotting facility was used to carry out the acoustic measurements as part of this study. It consists of a tank filled with degassed, deionised water of dimensions $0.7 \text{ m} \times 0.4 \text{ m} \times 0.5 \text{ m}$, over which is positioned a scanning rig. The rig has two independent carriages and mounts, each with longitudinal, vertical, transverse, angular and tilt adjusters, featuring ten degrees of freedom in total. One mount houses the emitting transducer and another the hydrophone. The linear resolution of the system is $2.5 \text{ } \mu\text{m}$ and the angular resolution 0.0036° . The scanning tank system is computer-controlled.

2.4 BERGEN CODE

2.4.1 Acquisition and implementation

The harmonic content of the hydrophone/amplifier output voltage waveform obtained at selected distances along the axis of symmetry of the 5 MHz focused transducer, is related to the harmonic content of the acoustic pressure in the propagating medium via the transfer characteristics of the hydrophone/amplifier system. In this section, it will be shown how the frequency content of the acoustic pressure waveform as a function of axial position was estimated using a finite difference (FD) algorithm based on the axisymmetric form of the KZK equation expressed in cylindrical coordinates. The KZK equation is a nonlinear parabolic equation in which nonlinearity and diffraction in sound beams is taken into account (Hamilton and Blackstock 1997), (Kuznetsov 1971). In the parabolic approximation, it is assumed that the energy propagates in a narrow beam. The KZK equation is therefore suitable for sources whose radial dimension encompasses several wavelengths and at field positions that are not too close to the source plane. Naze Tjøtta et al (1991) stipulate that, for focused sources, the following equations should be satisfied.

$$\frac{D}{a} \gg 1 \quad (4)$$

$$z \geq \frac{1}{k} \left(\frac{D}{a} \right)^{\frac{4}{3}} \quad (5)$$

where:

- D is the source radius of curvature
- a is the source radius
- z is the axial distance from the source
- k is the wavenumber in the propagating medium.

In this case, $D = 60$ mm and $a = 5$ mm. Given that comparison of the harmonic content of the hydrophone/amplifier voltage waveform with that of the acoustic pressure will be carried out at axial separation greater or equal to 60 mm, and at frequencies greater or equal to 5 MHz, the above equations are satisfied.

A pressure field where magnitude and phase of acoustic pressure do not vary too much with position is desirable, or at least an adequate position in field should be used to minimise positioning error.

The spatial component of the radial acoustic pressure distribution at the source was obtained using the following expression (Lucas and Muir 1982):

$$p(r,0) = j\rho_0\omega u_0 D \frac{\exp[jk(D^2 + r^2)]}{\sqrt{(D^2 + r^2)}} \quad (6)$$

where:

- ω is the angular frequency,
- u_0 is the piston velocity amplitude.

A source code for the axisymmetric FD implementation of the KZK equation for continuous wave excitation was located on the Internet (Universitetet i Bergen Website). Co-workers of Prof. Sigve Tjøtta and Prof. Jacqueline Naze Tjøtta developed the code during the 1980's and 1990's. Having been developed at the University of Bergen, Norway, this code is known as the Bergen code. Further details can be found in (Berntsen 1990).

After having downloaded the source code, it was of interest to modify it into a stand-alone executable file, which could be called from the DOS prompt, hence facilitating integration with MATLAB® script. Key input parameters detailing the source characteristics, the properties of the propagating medium as well as parameters pertaining to the numerical FD solution are listed into a standard text input file, which are then called from the executable. These are shown below.

- c_0 : the speed of sound,
- p_0 : the characteristic pressure amplitude of the source,
- a : the characteristic radius of the source,
- f_0 : the frequency of the source,
- ρ_0 : the density of the medium,
- δ : the sound diffusivity,
- β : the coefficient of nonlinearity,
- $stopz$: the maximum value of z that the calculation will run to,
- n_{out} : the number of output points of the radial distributions along the z axis $\{z_j, j = 1, 2, \dots, n_{out}\}$, the locations of the points on the axis at which the radial pressure distribution is required,
- N_{zout} : the number of points at which to output the on-axis results,
- z_{min} : the minimum value of z at which the on-axis values are required,
- z_{max} : the maximum value of z at which on-axis values are required,
- M : the number of harmonics used in the summation, maximum of 1000,
- n_r : the number of radial measurements used in the boundary conditions, $\{r_i, \text{Re}\{F(r_i)\}, \text{Im}\{F(r_i)\}, i = 1, 2, \dots, n_r\}$, the measurements for the boundary conditions,
- r_{max} : the size of the domain in the r direction at $z = 0$,
- N_r : the number of points to use in the radial direction (max 1000, more points improves results quality but increases run time, recommended min 255),
- dz : the main step size in the axial direction, in the same units as all other lengths.

The acoustic pressure field data is then printed in a series of output text files. Further detail on the equations that the software solves may be found in Annex A. The software manual of the most recent version of the code (v7) is located in Annex B.

2.4.2 Use of Bergen code to predict on-axis pressure field of 5 MHz focused transducer

An outline of the method used by Cooling and Humphrey (2006, 2007) is shown below.

- A circular, well-behaved focused transducer is used.

- Uniform amplitude variation across the face is assumed and the effective radiating diameter and focal length are adjusted to give an optimal fit with the measured fundamental behaviour obtained from an axial scan.
- Comparison of hydrophone output voltage and acoustic pressure obtained using the Bergen code is carried out beyond the focus, to minimise effects of spatial averaging due to finite hydrophone element size, and to ensure that the magnitude and phase of the acoustic pressure within the field vary as little as possible so as to minimise positioning error.
- The Bergen code, being a FD solution to the KZK equation, includes the effects of diffraction.
- The pulses used experimentally were tone bursts, i.e. quasi-continuous wave.
- The model was run for continuous wave.

An outline of the method used by Bleeker and Lewin (2000) is shown below.

- The KZK equation is solved for pulsed mode, not continuous wave.
- Uniform distribution of pressure amplitude and phase velocity across the source surface is assumed.
- The source aperture, radius of curvature and pressure amplitude at the transducer's surface are measured under linear propagation conditions.
- These values then enable the KZK model source input parameters to be estimated through minimising the least squares error between the simulated and measured axial pressure distribution
- Spatial averaging effects are compensated for by calculating the pressures at points along the effective hydrophone radius and integrating over the effective hydrophone area.

The methodology employed as part of this feasibility study differed slightly from the two cases above in the details described below.

- Prior knowledge of the magnitude of the hydrophone/amplifier transfer characteristics was assumed between 5 MHz and 40 MHz, obtained using a primary laser interferometer calibration.
- The acoustic pressure magnitude of the first n harmonics was estimated from knowledge of voltage waveforms acquired at selected axial positions measured under high drive conditions and knowledge of the hydrophone/amplifier combined sensitivity. Educated guesses for the initial values of u_0 , a , D , β and δ were used.
- The acoustic pressure source distribution was calculated using equation (6).
- The Bergen code was then used to forward-propagate the above source distribution to the axial measurement positions.
- u_0 , a , D , β and δ were then varied using the simplex search method described by Lagarias et al (1998), and the above two steps were repeated until a local minimum

was found for $\sum_{i=1}^m \sum_{j=1}^n \left| w_i (P_{KZK}^{i,j} - P_{meas}^{i,j}) \right|^2$, where

- P_{KZK} is the pressure magnitude predicted by the KZK algorithm;
- P_{meas} is the measured pressure magnitude;
- i is the index corresponding to the axial position;
- j is the frequency index;

- m is the total amount of measurement positions considered as part of the optimisation;
- n is the total amount of harmonics considered as part of the optimisation;

and

$$w_i = \frac{1}{\max(P_{KZK}^i)}$$

where $\max$ denotes the maximum value and P_{KZK}^i is the axial pressure magnitude of the i^{th} harmonic.

$\max(P_{KZK}^i)$ in fact corresponds to the value of the last axial maximum of the i^{th} harmonic, given the axial positions selected.

- The combined sensitivity, S_{KZK} , of the hydrophone/amplifier system was then estimated over the desired frequency range from the ratio of the measured voltage magnitude and the on-axis pressure obtained using the KZK algorithm at each harmonic:

$$S_{KZK}^{i,j} = \frac{V_{meas}^{i,j}}{P_{KZK}^{i,j}} \quad (7)$$

where $V_{meas}^{i,j}$ corresponds to the measured hydrophone/amplifier output voltage magnitude of the j^{th} harmonic at the i^{th} measurement position.

The total amount of harmonics n retained during the optimisation process will relate to the value of the harmonic of the fundamental (i.e. 5 MHz), which corresponds to the upper limit of the calibration frequency range. As this is 40 MHz, $n = 8$.

The FORTRAN executable was called from a MATLAB® script file. Once the final values for D , a , u_0 , β and δ were obtained, the resulting pressure distribution at the source predicted using equation (6) was then forward-propagated using the Bergen code. This provided the magnitude and phase of each harmonic component of the on-axis acoustic pressure at the chosen axial distances. The hydrophone output voltages were then compared with the simulated acoustic pressure at separations large enough so that there was sufficient harmonic content at the upper frequency limit of the investigation. Comparison of the hydrophone-measured open-circuit output voltage with the predicted pressure field may otherwise not have yielded reliable results. Nevertheless, data at smaller transducer-hydrophone separations was retained in the optimisation process, given that information regarding the location of the last axial minimum is important in characterising the field.

Starting values of D , a , u_0 , β and δ were used, and the predicted on-axis pressure magnitude was compared against the measured pressure between 5 MHz and 40 MHz, based on hydrophone output voltage measurements and knowledge of the hydrophone sensitivity obtained using interferometer primary calibration data. Spatial averaging effects were accounted for by integrating the acoustic pressure over the hydrophone element area. As the field is assumed to be axisymmetric, this is equivalent to evaluating the following integral at each transducer-hydrophone separation,

$$\frac{2}{a^2} \int_0^a r p(r, z) dr \quad (8)$$

where:

- a is the radius of the hydrophone active element,
- r is the radial distance,
- z is the axial distance,
- $p(r, z)$ represents the spatial component of the acoustic pressure.

This was carried out numerically by considering the radial distribution of the acoustic pressure at specified values of z .

The above procedure was repeated using the Nelder-Mead simplex (direct search) method until the least squares difference was minimised. Only 5% variation was tolerated on literature values of β and δ , as stated in (Hamilton and Blackstock, 1997).

The magnitude and relative phase of the hydrophone transfer characteristics were then estimated at each position, and a mean value obtained between 5 MHz and 100 MHz in 5 MHz steps. A combination of extrapolation and interpolation may be used to estimate the response below 5 MHz and at a resolution finer than 5 MHz.

3. RESULTS

3.1 TRANSDUCER SCANS

Using the NPL Beam Plotting Facility described in Section 2.3, scans were carried out to gain further insight into the ultrasonic field produced by the transducer so as to assess its suitability for this feasibility study.

An Agilent® 33250A Function Generator was used to generate a 50 mV peak-to-peak, six-cycle tone-burst of 5 MHz fundamental frequency. This was amplified using a 50 dB ENI®150 power amplifier. The amplifier was used to drive a V326 5 MHz Panametrics® focused transducer, of 0.375" (9.525 mm) diameter with nominal focal length 60 mm.

The acoustic field was characterised using IP901 (see Section 2.2 for specifications) into a 150 MHz broadband Comlinear® amplifier. The amplifier was connected to a Tektronix® TDS784D oscilloscope via a 50 Ω shunt resistor.

An axial scan was carried out by acquiring the RMS output voltage of the hydrophone/amplifier measurement system at transducer-hydrophone separations ranging from 6 mm to 186 mm in steps of 0.2 mm. Results are shown in Fig. 3.

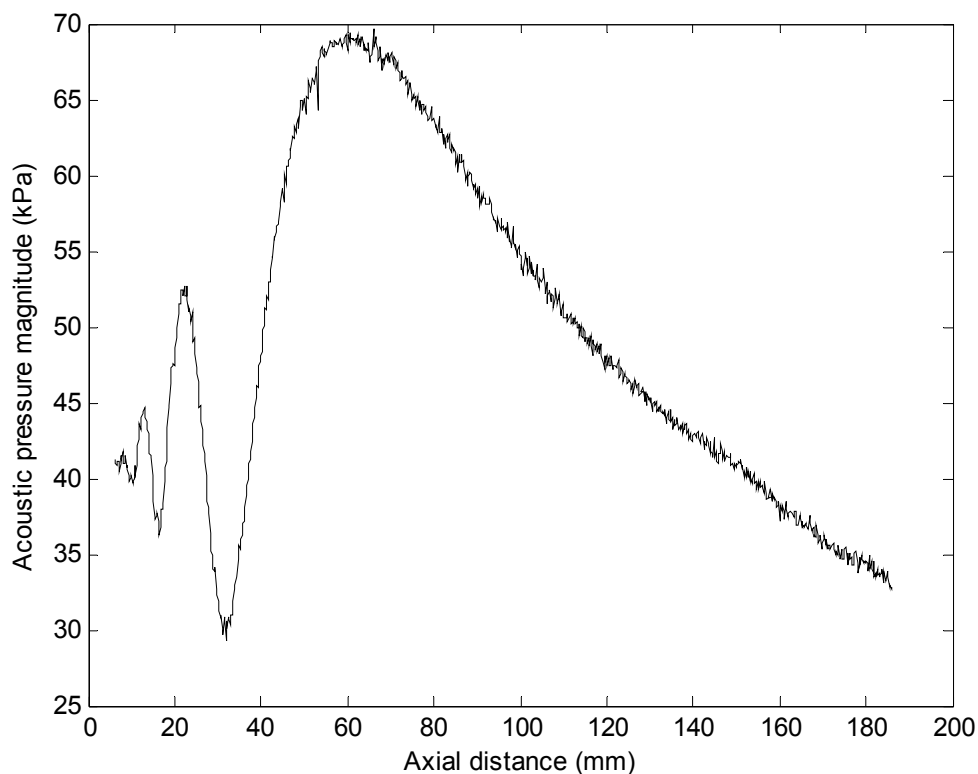


Fig. 3 – Axial scan of the Panametrics® V326 5 MHz transducer using IP901 into a 150 MHz broadband Comlinear® amplifier.

A raster scan was then completed at 6 mm away from the front-face of the transducer using the above instrumentation. A square grid was chosen ranging from -7 mm to $+7$ mm from the axis of symmetry in 2 mm steps. Results are shown in Fig. 4.

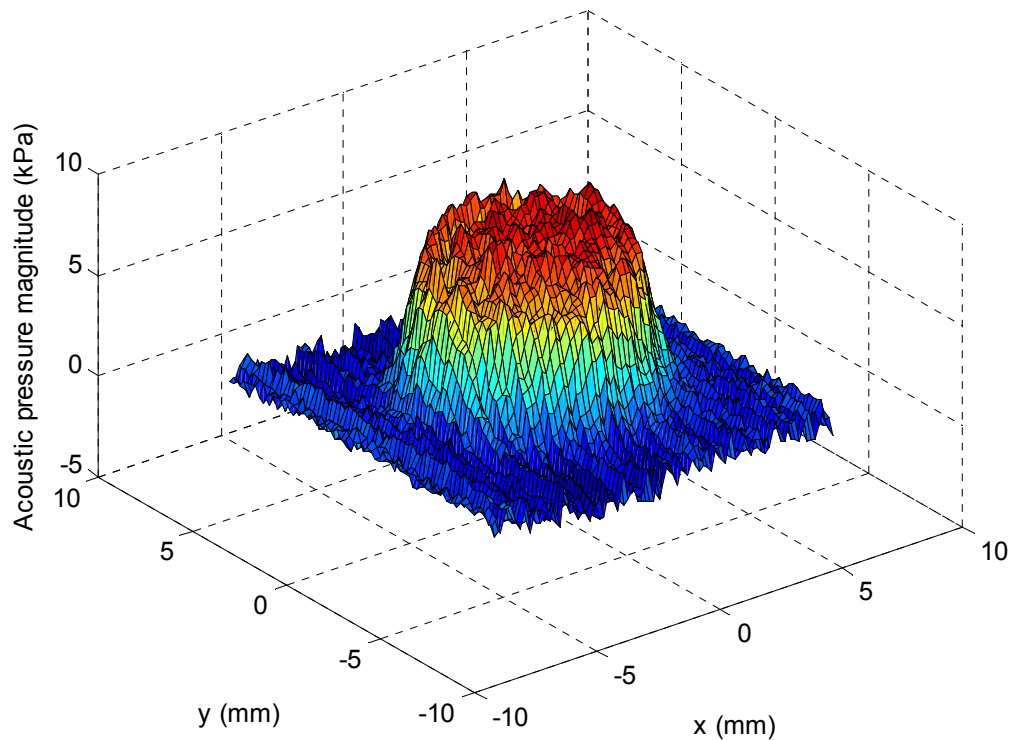


Fig. 4 – Raster scan of Panametrics® V326 5 MHz transducer using using IP901 into a 150 MHz broadband Comlinear® amplifier.

The hydrophone/amplifier output voltage, which is proportional to the acoustic pressure magnitude distribution across the transducer front face, is reasonably circularly symmetric. The pressure magnitude is approximately constant over a 4 mm radius and the source appears close to a top-hat distribution.

3.2 REFERENCE HYDROPHONE CALIBRATION RESULTS

The reference broadband hydrophone used as part of this study was calibrated using laser interferometry. The procedure employed was the same as that described by Esward and Robinson (1999), except for minor changes in the instrumentation. The signal generator was an Agilent® 33250A Function Generator providing a 35 cycle 5 MHz 350 mV gated tone-burst signal. This was then amplified with an AR® power amplifier. The hydrophone-transducer separation was 60 mm.

The average of three calibrations is taken. The results are shown in Fig. 5.

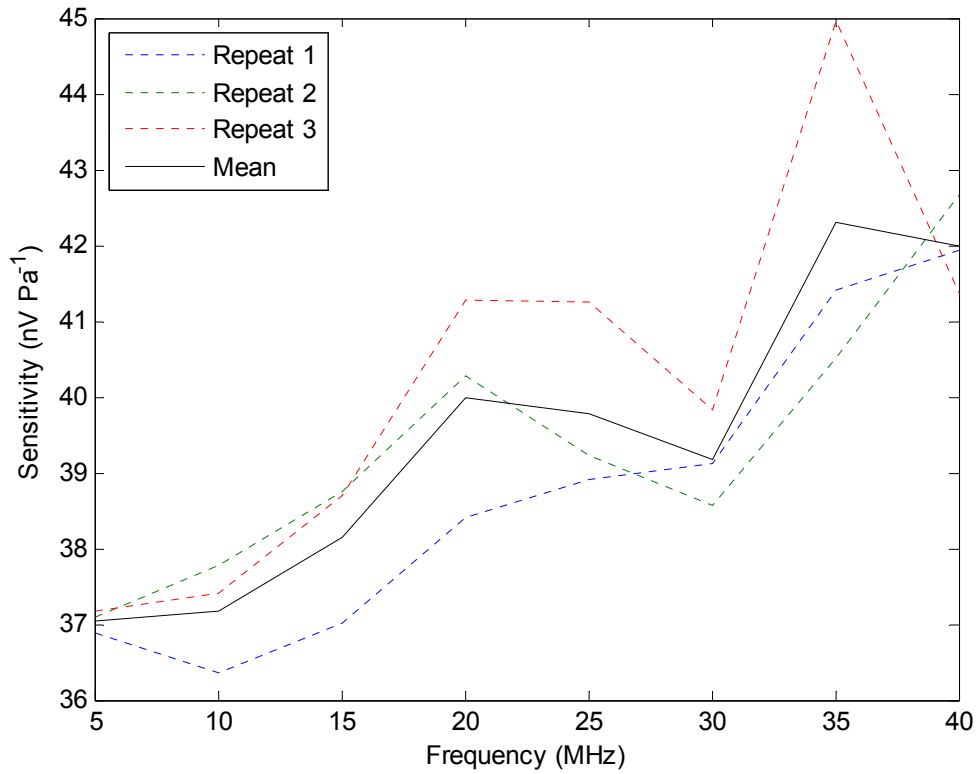


Fig. 5 – Primary standard interferometer calibration of IP904 obtained from three repeats.

3.3 AMPLIFIER GAIN AND LOADING CORRECTION

In order to obtain the open-circuit hydrophone sensitivity from the loaded sensitivity, the factor $T(\omega)$ needs to be evaluated, as defined in equation (3). This necessitates measuring the following quantities.

- Hydrophone end-of-cable electrical impedance Z_{hyd} , with hydrophone soaked in water;
- Measurement of Onda® AH-2010-100 amplifier input impedance on high gain setting, Z_{in} ;
- Measurement of Onda® AH-2010-100 amplifier complex gain, $G(\omega)$.

The electrical impedance quantities Z_{hyd} and Z_{in} are measured between 0.5 MHz and 100 MHz in 0.5 MHz steps using an Agilent® 4294A impedance analyser. The amplifier gain is measured using a Hewlett Packard® 3589A Spectrum Analyser. Electrical impedance data is shown in Figs. 6-9.

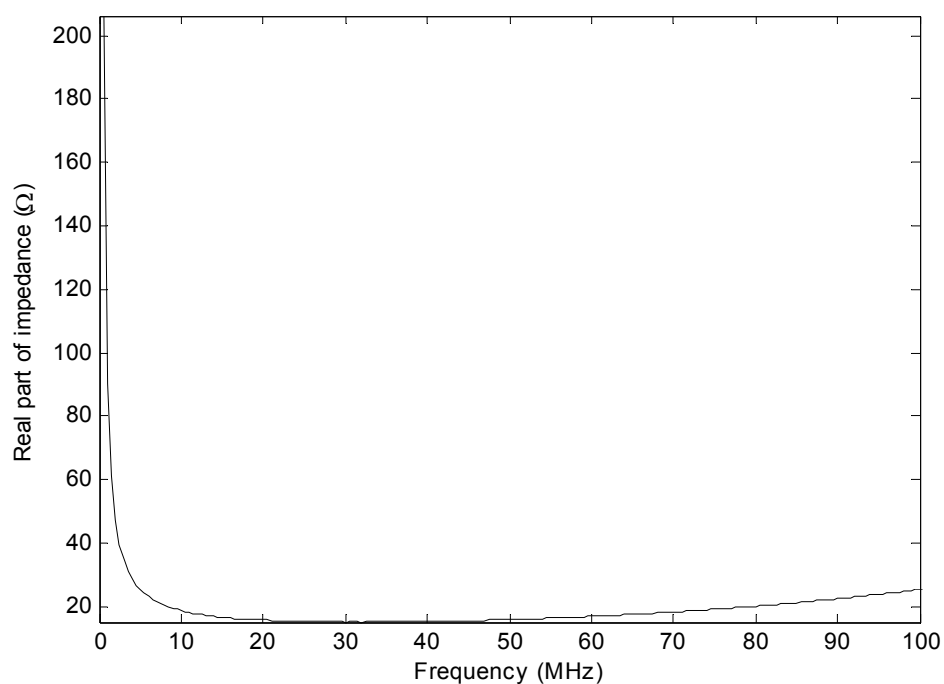


Fig. 6 – Real part of end-of-cable electrical impedance of IP904 (soaked).

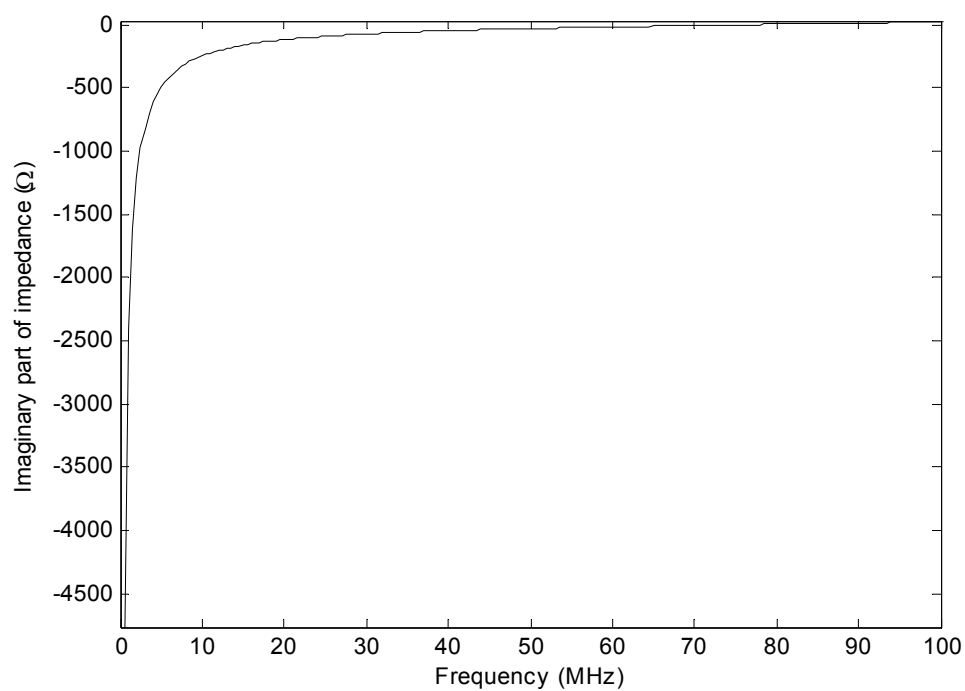


Fig. 7 – Imaginary part of end-of-cable electrical impedance of IP904 (soaked).

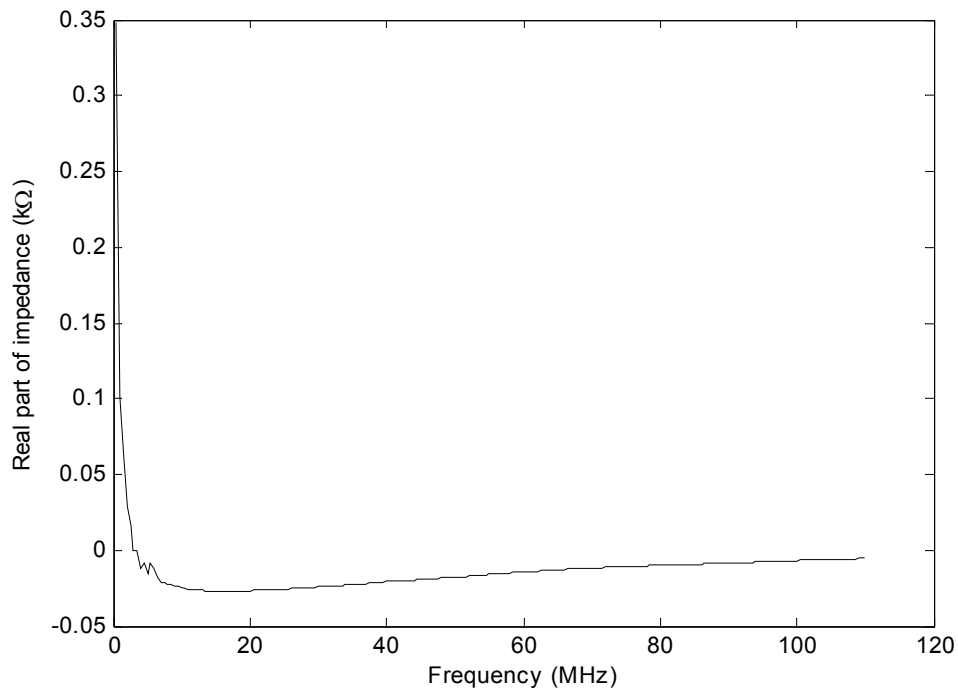


Fig. 8 – Real part of electrical impedance of input stage of Onda® AH-2010-100 amplifier on high-gain setting.

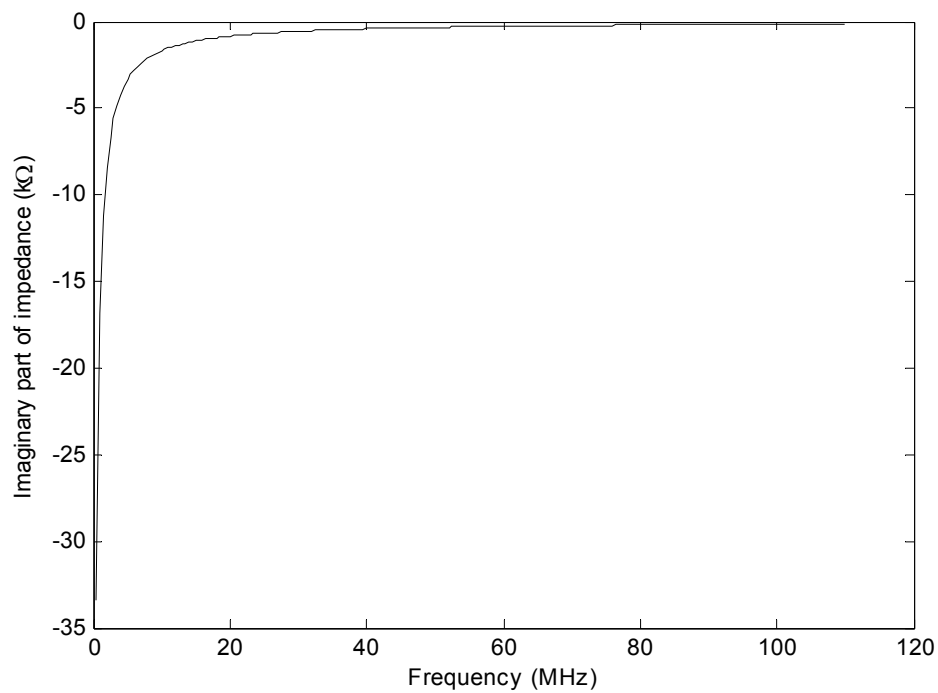


Fig. 9 – Imaginary part of electrical impedance of input stage of Onda® AH-2010-100 amplifier on high-gain setting.

The real and imaginary parts of the voltage gain of the Onda® amplifier are shown in Figs. 10-11. The voltage gain in dB is displayed in Fig. 12.

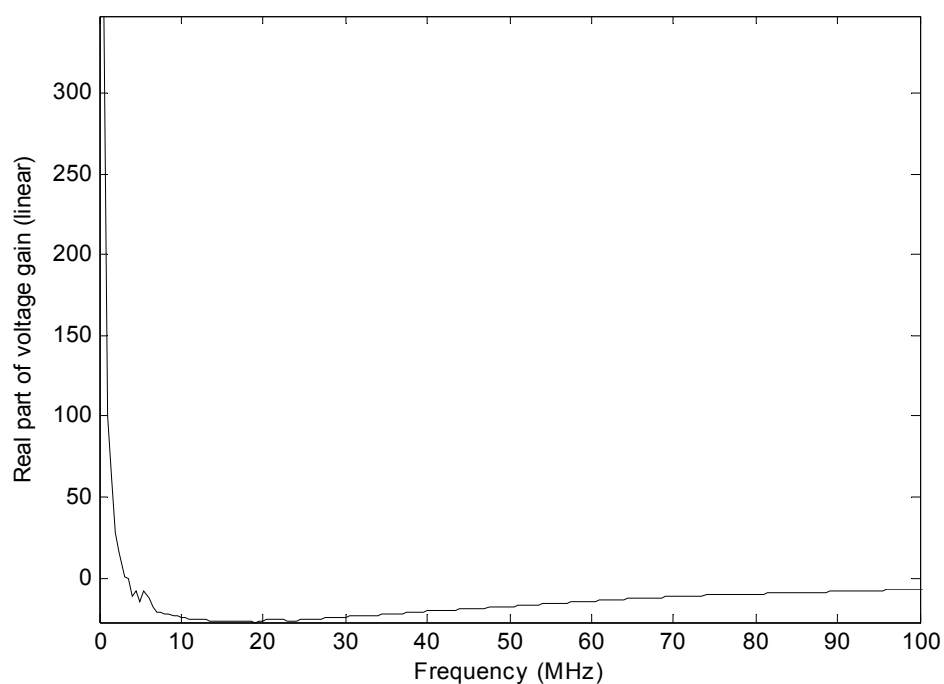


Fig. 10 – Real part of voltage gain of Onda® AH-2010-100 amplifier on high-gain setting.

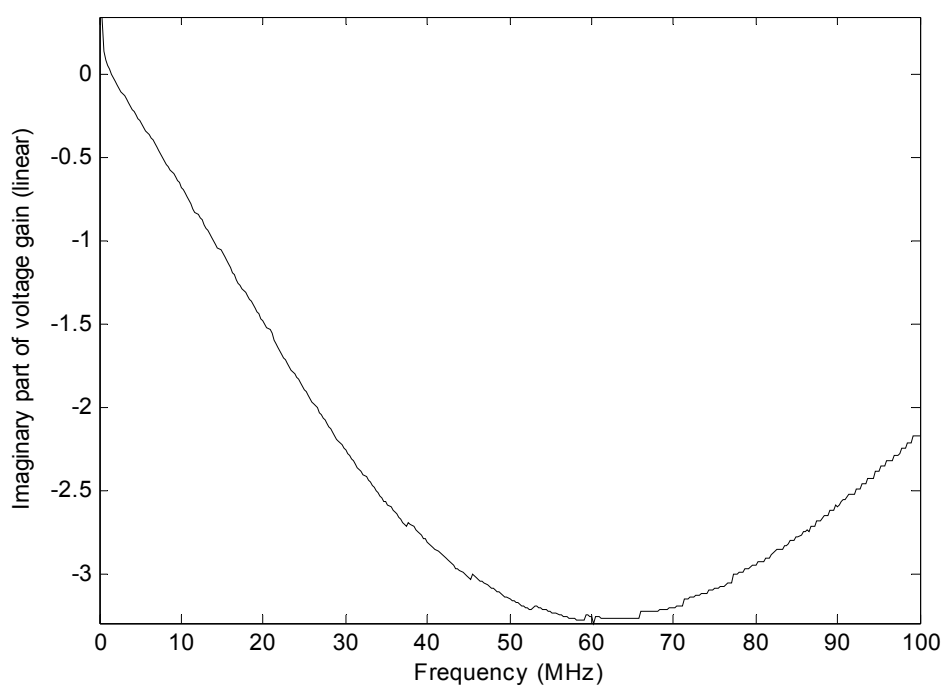


Fig. 11 – Imaginary part of voltage gain of Onda® AH-2010-100 amplifier on high-gain setting.

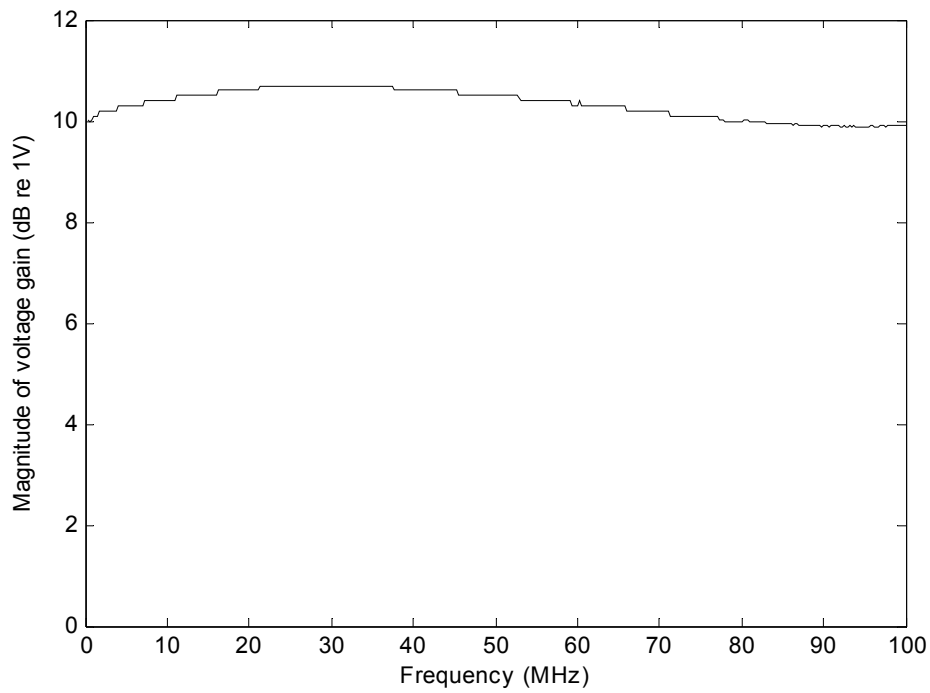


Fig. 12 – Magnitude of voltage gain of Onda® AH-2010-100 amplifier on high-gain setting in dB.

It can be seen in Fig. 12 that the voltage gain of the Onda® amplifier is flat within ± 0.5 dB up to 100 MHz.

The real and imaginary parts of the electrical loading factor T as a function of frequency are displayed in Figs. 13-14.

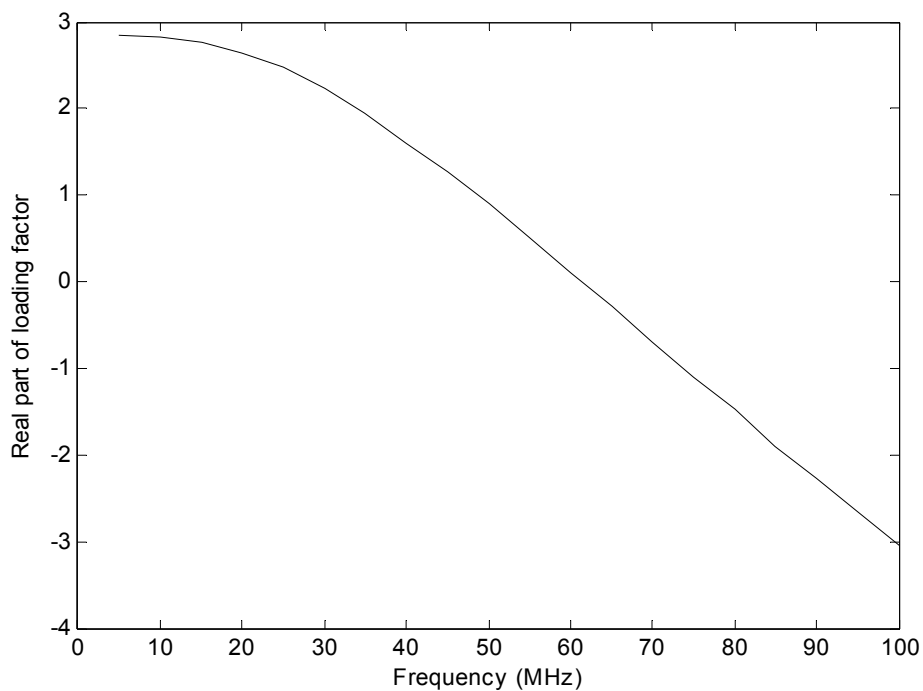


Fig. 13 – Real part of electrical loading factor T , as defined in equation (3).

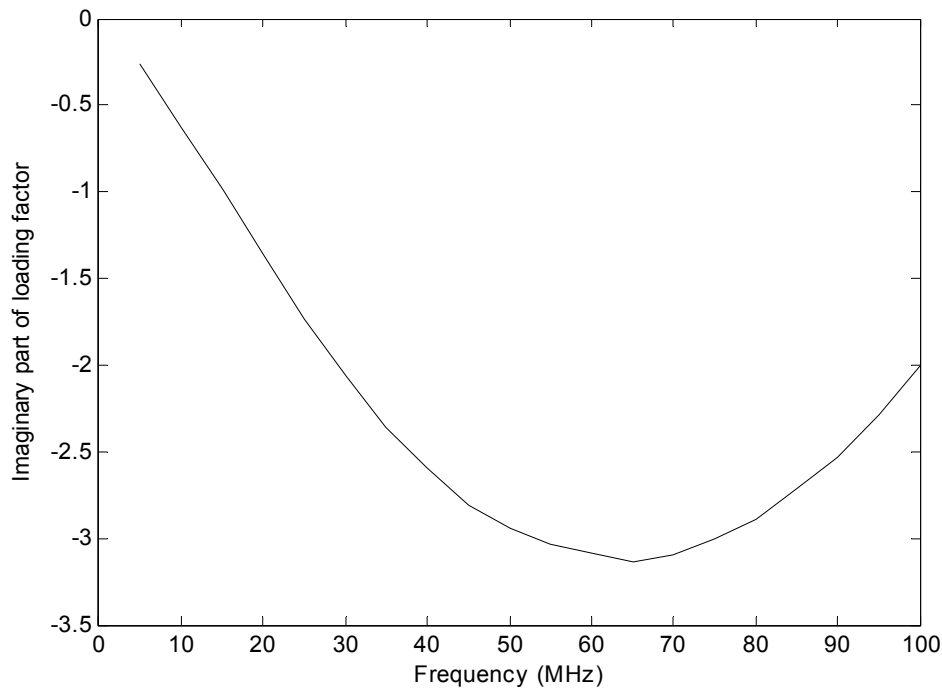


Fig. 14 – Imaginary part of electrical loading factor T , as defined in equation (3).

3.4 AXIAL SCANS

The voltage associated with each harmonic component between 5 MHz and 100 MHz at the amplifier output stage was investigated as a function of transducer-hydrophone axial separation.

The V326 Panametrics® 5 MHz focused transducer was driven with an Agilent® 33250A Function Generator providing a 15 cycle, 5 MHz, 450 mV (peak-to-peak) gated tone-burst waveform, amplified with a ENI® 325-LA 50 dB power amplifier. One hundred single shot waveforms were acquired at each axial position using IP904 connected to the Onda® AH-2010-100 broadband amplifier on high-gain setting into a Tektronix TDS784D oscilloscope.

One hundred waveforms were acquired at each of the following separations:

- 40 mm
- 50 mm
- 60 mm
- 70 mm
- 80 mm
- 90 mm
- 100 mm
- 120 mm
- 140 mm
- 160 mm
- 180 mm.

This series of measurements was carried out at a temperature of 19.3°C. Four cycles were extracted after initial transients had decayed (10th to 13th cycle), as shown in Fig. 15, and a single cycle was obtained through averaging, after aligning the 100 waveforms to the nearest zero crossing of the 11th cycle.

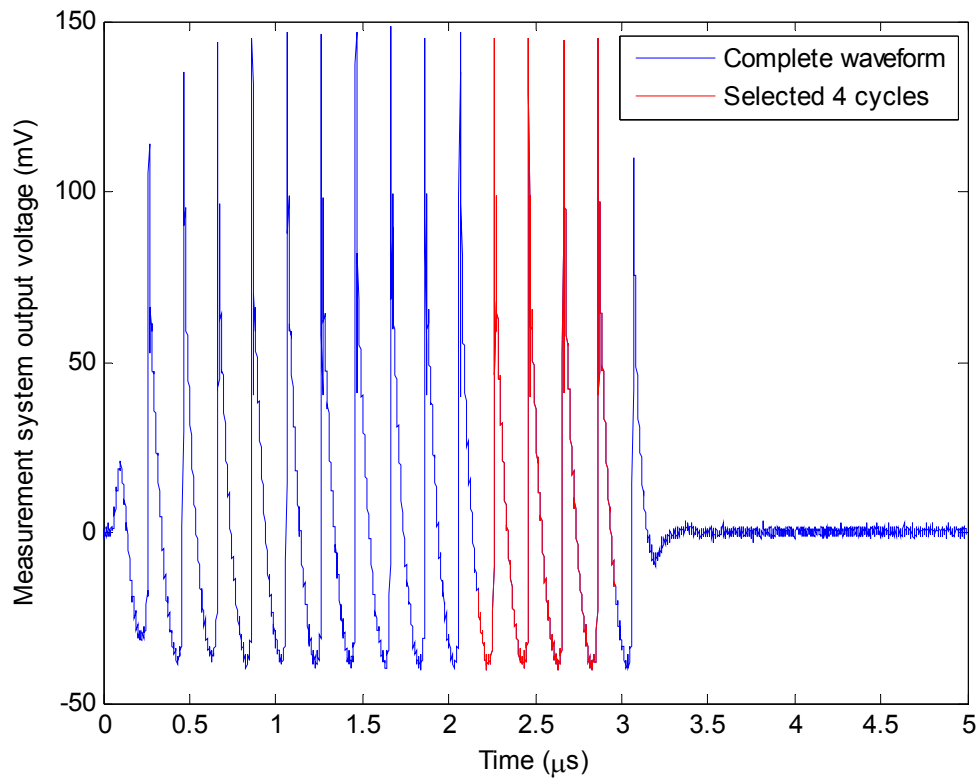


Fig. 15 – Single acquisition of IP904/Onda® AH-2010-100 amplifier output voltage, measuring the acoustic field produced by V326 Panametrics® 5 MHz focused transducer at 100 mm from source.

A Fourier series for this single cycle was then computed, hence giving magnitude and phase information of the hydrophone output voltage at each separation. Information up to and including 100 MHz is retained. Results are shown in Figs. 16-18.

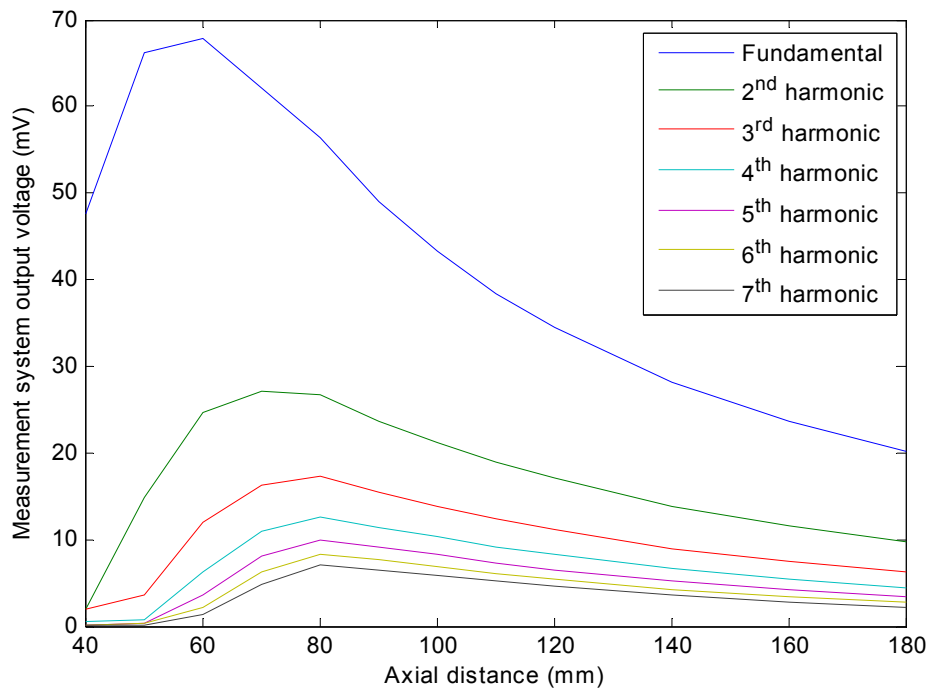


Fig. 16 – Harmonic content of IP904/Onda® AH-2010-100 amplifier output voltage as a function of axial position, measuring the acoustic field produced by V326 Panametrics® 5 MHz focused transducer.

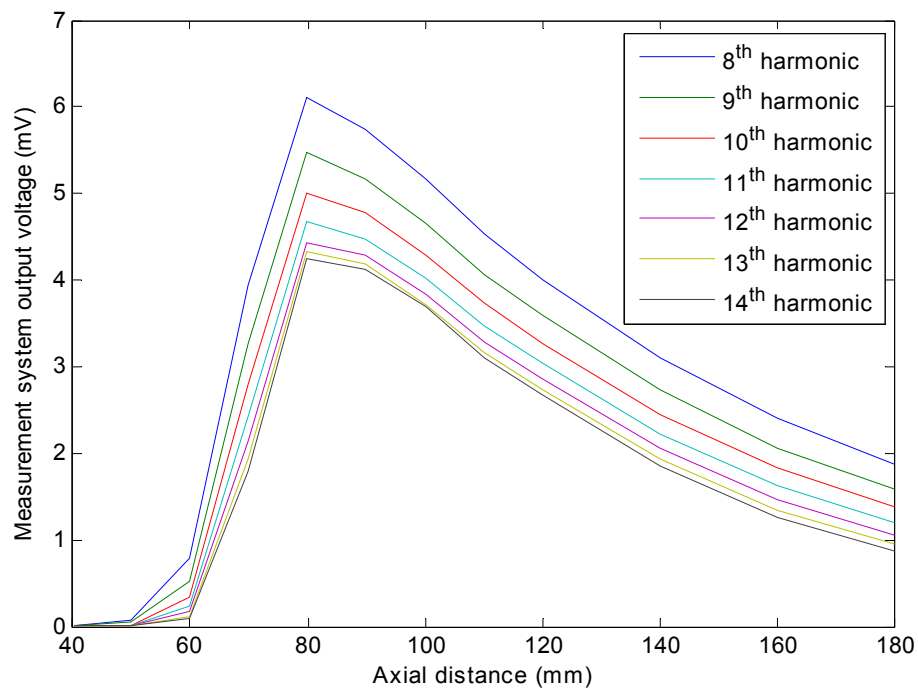


Fig. 17 – Harmonic content of IP904/Onda® AH-2010-100 amplifier output voltage as a function of axial position, measuring the acoustic field produced by V326 Panametrics® 5 MHz focused transducer; 8th to 14th harmonic.

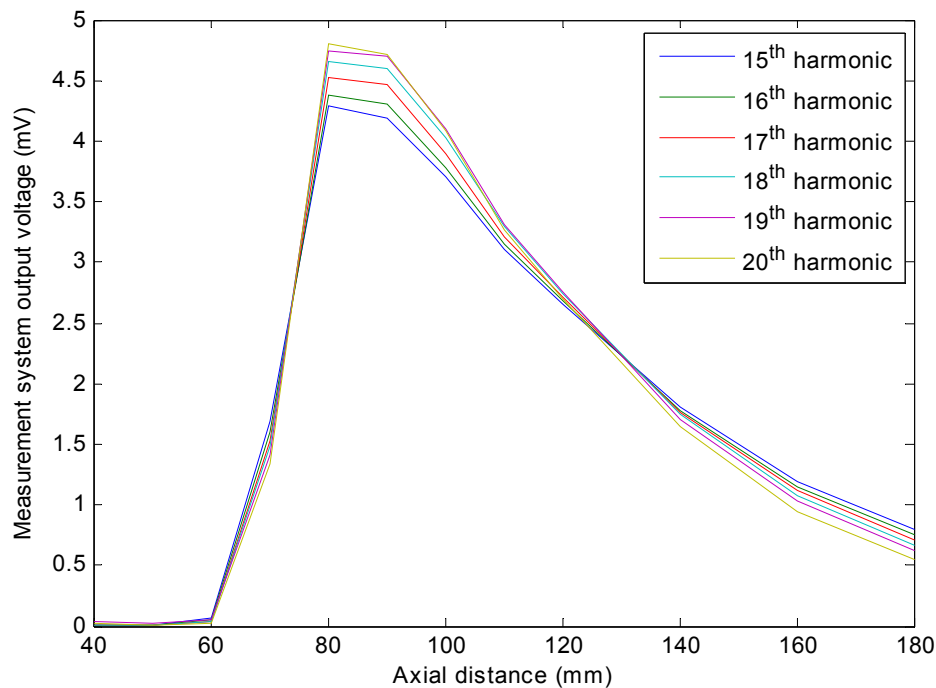


Fig. 18 – Harmonic content of IP904/Onda® AH-2010-100 amplifier output voltage as a function of axial position, measuring the acoustic field produced by V326 Panametrics® 5 MHz focused transducer; 15th to 20th harmonic.

The results in Figs. 16-18 show that the 20th harmonic component is 20 dB down relative to the fundamental at 90 mm away from the transducer hence demonstrating that there is significant frequency content at 100 MHz.

3.5 VALIDATION OF BERGEN CODE: NPL-ISVR INTERCOMPARISON

In order to help validate the results of the Bergen code implemented at NPL from the FORTRAN source code obtained from (Universitetet i Bergen Website), it was desirable to set up some benchmark tests so that the code could be validated against that of another research institute.

Contact was established with Martin Cooling (ISVR, University of Southampton, UK) and some initial tests were run which, although showing similar trends around the focus and fairly good far-field agreement, did not produce results that were deemed to be in satisfactory agreement. In the original formulation of the code used at NPL, an implicit method was used for the first few steps followed by a longer step size in conjunction with an explicit formulation. This was discarded in favour of using an implicit solution throughout, which has the advantage of producing improved results, albeit at the expense of longer run times.

Once these issues were addressed, together with ensuring the same input parameters were used in the NPL and ISVR versions of the Bergen code, a low-drive run (i.e. no higher order harmonics) for an unfocused plane circular piston source was carried out. The following input parameters were used (see Fig. 19).

- On-axis pressure magnitude at source: $p_0 = 1$ Pa.
- Source radius: $a = 4$ mm.
- Fundamental frequency: $f_0 = 5$ MHz.
- Diffusivity of sound: $\delta = 4.12545002 \times 10^6$.
- Nonlinear parameter: $\beta = 3.45$.
- Axial step size: $\pi a^2 f_0 / c_0 \times 10^{-5}$.
- Sound speed: $c_0 = 1482.36$ m s⁻¹.
- Equilibrium density: $\rho_0 = 998.2$ kg m⁻³.
- Number of points along source: 91.
- Number of points along radial boundary: 801.
- Number of harmonics retained in the solution: 10.

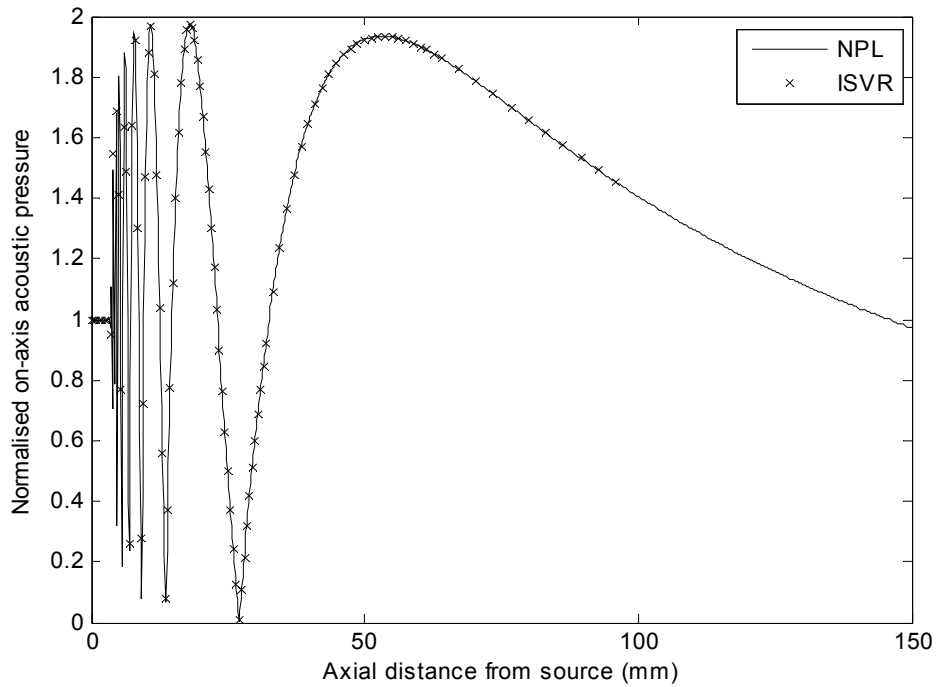


Fig. 19 – NPL/ISVR comparison of axial acoustic pressure output from Bergen code for a 5 MHz, 4 mm radius plane piston source.

Another low-drive comparison was carried out, this time using a focused source. The following input parameters were used (see Fig. 20).

- On-axis pressure magnitude at source: $p_0 = 0.122 \times 10^{-5} \times (\rho_0 c_0)$ Pa.
- Source radius: $a = 5$ mm.
- Source radius of curvature: $D = 80$ mm.
- Fundamental frequency: $f_0 = 5$ MHz.
- Diffusivity of sound: $\delta = 4.12545002 \times 10^6$.
- Nonlinear parameter: $\beta = 3.45$.
- Axial step size: $\pi a^2 f_0 / c_0 \times 10^{-5}$.
- Sound speed: $c_0 = 1482.36$ m s⁻¹.
- Equilibrium density: $\rho_0 = 998.2$ kg m⁻³.
- Number of points along source: 91.

- Number of points along radial boundary: 601.
- Number of harmonics retained in the solution: 10.

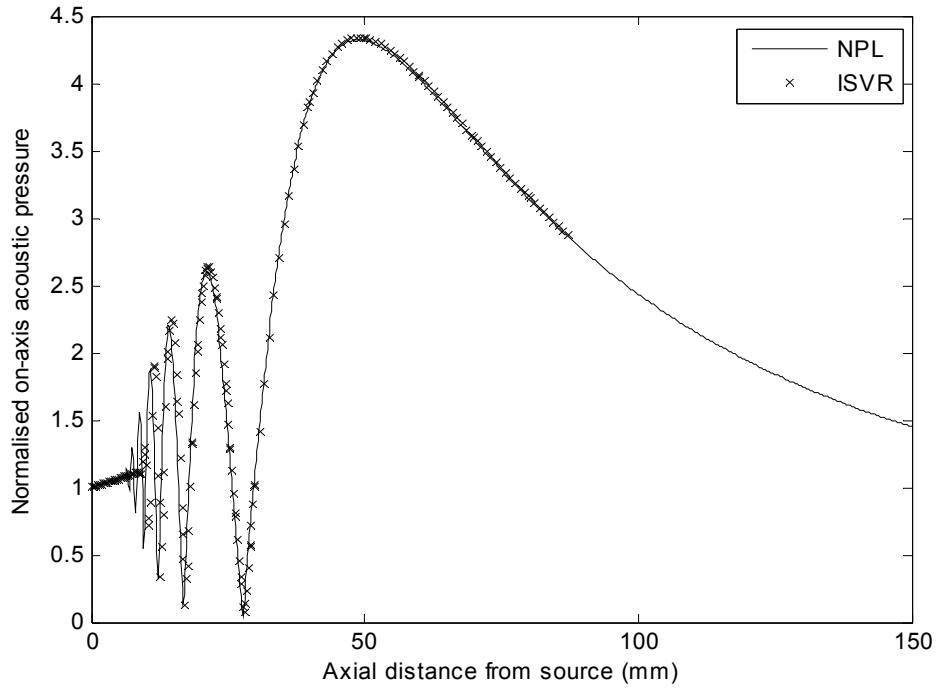


Fig. 20 – NPL/ISVR comparison of axial acoustic pressure output from Bergen code for a 5 MHz, 5 mm radius focused piston source, with radius of curvature 80 mm. Low drive.

A high-drive run for the above focused piston source was also carried out, with an on-axis pressure magnitude at source of $p_0 = 0.122/(\rho_0 c_0)$ Pa. 40 harmonics were retained in the solution (see Figs. 21 and 22).

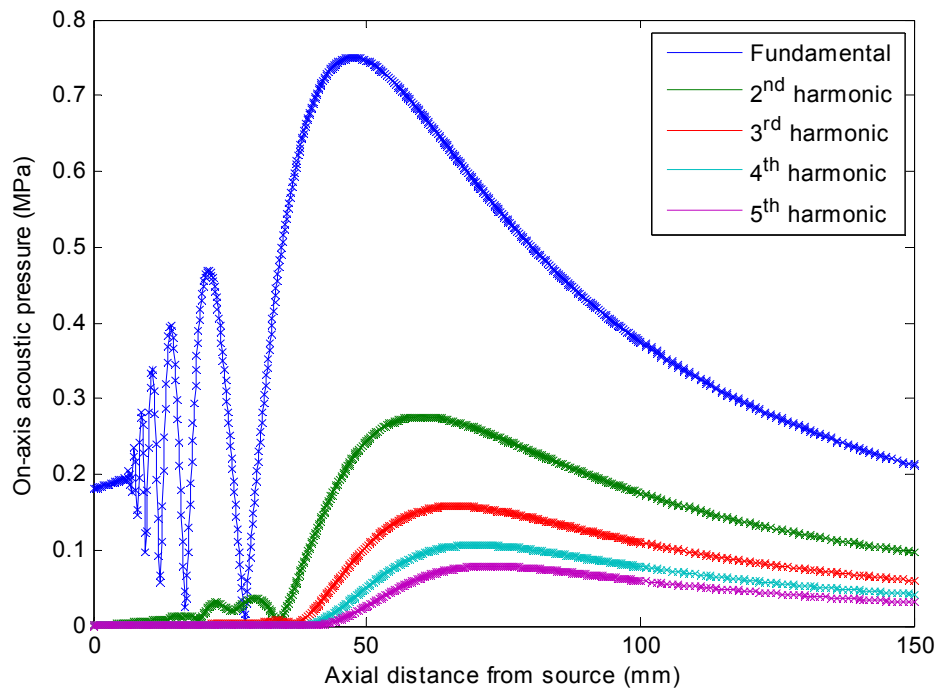


Fig. 21 – NPL/ISVR comparison of axial acoustic pressure output from Bergen code for a 5 MHz, 5 mm radius focused piston source, with radius of curvature 80 mm. High amplitude. Harmonics 1-5. —: NPL. ×: ISVR.

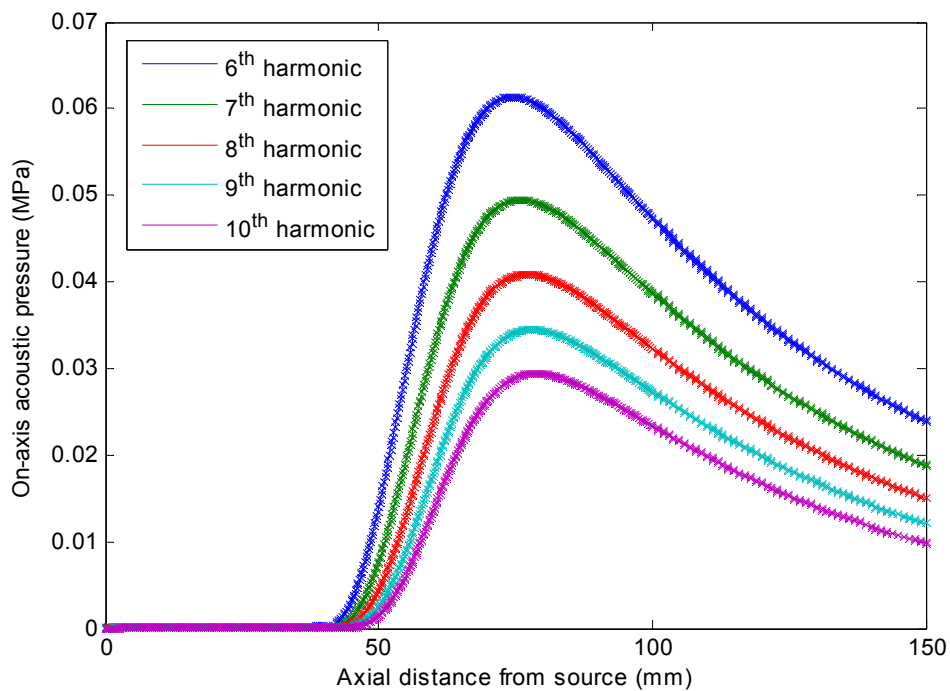


Fig. 22 – NPL/ISVR comparison of axial acoustic pressure output from Bergen code for a 5 MHz, 5 mm radius focused piston source, with radius of curvature 80 mm. High amplitude. Harmonics 6-10. —: NPL. ×: ISVR.

Overall, excellent agreement is obtained between the NPL and ISVR implementations of the Bergen code for both planar and focused sources at low drive levels, where wave propagation is linear, and high drive levels where energy from the fundamental is

transferred to higher order harmonics. In the case of the focused source, for high drive levels, the agreement is within $\pm 0.5\%$ up to and including the 10th harmonic.

3.6 BERGEN CODE RESULTS

The initial values of the objective function variables are listed as follows.

- Source radius: $a = 5$ mm.
- Source radius of curvature: $D = 80$ mm.
- Diffusivity of sound: $\delta = 4.12545002 \times 10^{-6}$.
- Nonlinear parameter: $\beta = 3.45$.
- On-axis pressure magnitude at source: $p_0 = 0.122 \times (\rho_0 c_0)$ Pa.

Other input parameters for the Bergen code are listed below.

- Fundamental frequency: $f_0 = 5$ MHz.
- Sound speed: $c_0 = 1480.19$ m s⁻¹.
- Equilibrium density: $\rho_0 = 998.35$ kg m⁻³.
- Initial z value: 0 mm.
- Final z value: 180 mm.
- Axial step size: $2\pi a^2 f_0 / c_0 \times 10^{-4}$.
- Number of points along source: 71.
- Number of points along radial boundary: 501.
- Number of harmonics retained in the solution during optimisation: $M = 20$.
- Number of harmonics that predicted pressure field is optimised to: $n = 8$.

During the optimisation procedure described in Section 2.4.2, the radial distribution was obtained at each of the measurement positions listed in Section 3.4, and a spatial average of the predicted acoustic pressure over the hydrophone receiving area was calculated using equation (8). This value was compared against the measured value of the acoustic pressure magnitude at each axial position, until a local minimum for

$\sum_{i=1}^m \sum_{j=1}^n \left| w_i \left(P_{KZK}^{i,j} - P_{meas}^{i,j} \right) \right|^2$ was obtained. The final values of the objective function variables are as follows.

- Source radius: $a = 5.18$ mm.
- Source radius of curvature: $D = 107$ mm
- Diffusivity of sound: $\delta = 4.26 \times 10^{-6}$.
- Nonlinear parameter: $\beta = 3.62$.
- On-axis pressure magnitude at source: $p_0 = 0.124 \times (\rho_0 c_0)$ Pa.

Using the final values of a , D , δ , β and p_0 , the Bergen code was then re-run, this time retaining $M = 100$ harmonics in the solution, so as to ensure sufficient convergence up to 100 MHz.

Agreement of the predicted pressure field for harmonic content between 5 MHz and 40 MHz (i.e. the first 8 harmonics) and with scanning tank hydrophone measurements corrected for hydrophone/amplifier sensitivity is displayed in Figs. 23 and 24.

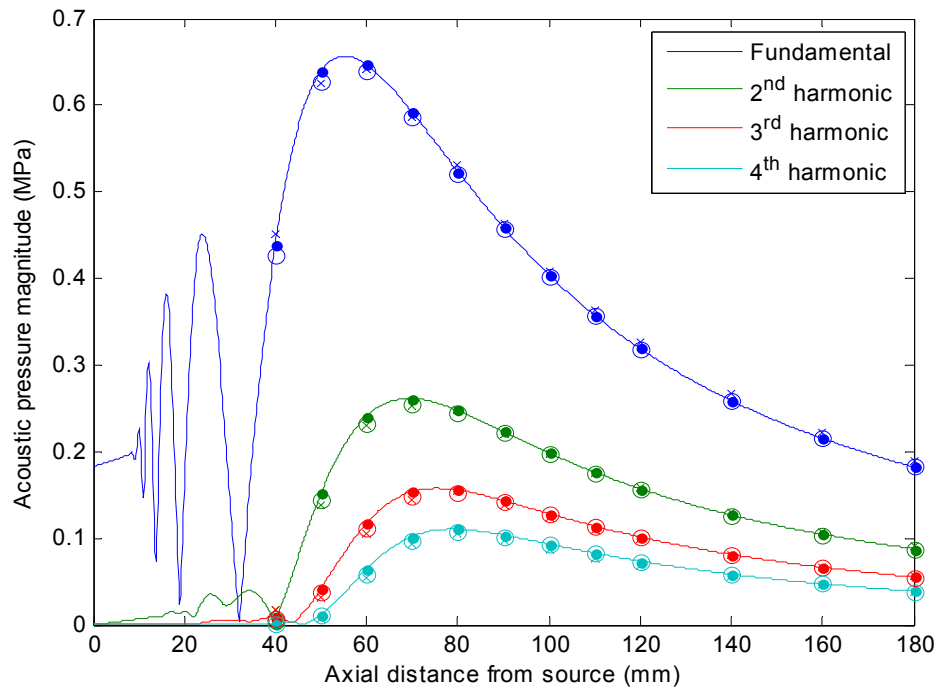


Fig. 23 – Bergen code field predictions with source parameters optimised to fit scanning tank measurement data. Harmonics 1-4 (respectively blue, green, red, cyan). Dots: predictions at measurement positions, crosses: measured data, circles: predictions with spatial averaging correction.

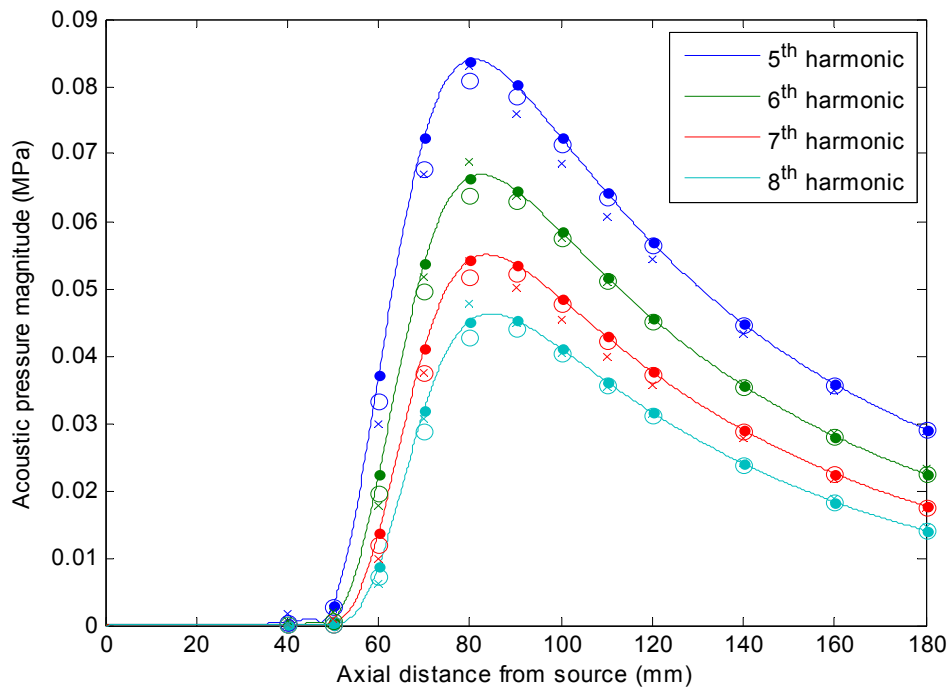


Fig. 24 – Bergen code field predictions with source parameters optimised to fit scanning tank measurement data. Harmonics 5-8 (respectively blue, green, red, cyan). Dots: predictions at measurement positions, crosses: measured data, circles: predictions with spatial averaging correction.

Radial variations of the pressure magnitude for the first 8 harmonics at three selected axial positions (60 mm, 90 mm and 180 mm) are shown in Figs. 25-27. Note that cubic spline interpolation was used to obtain this data, given that there is little variation of the pressure magnitude as a function of the radial coordinate, and that using more radial points would have resulted in extended run times.

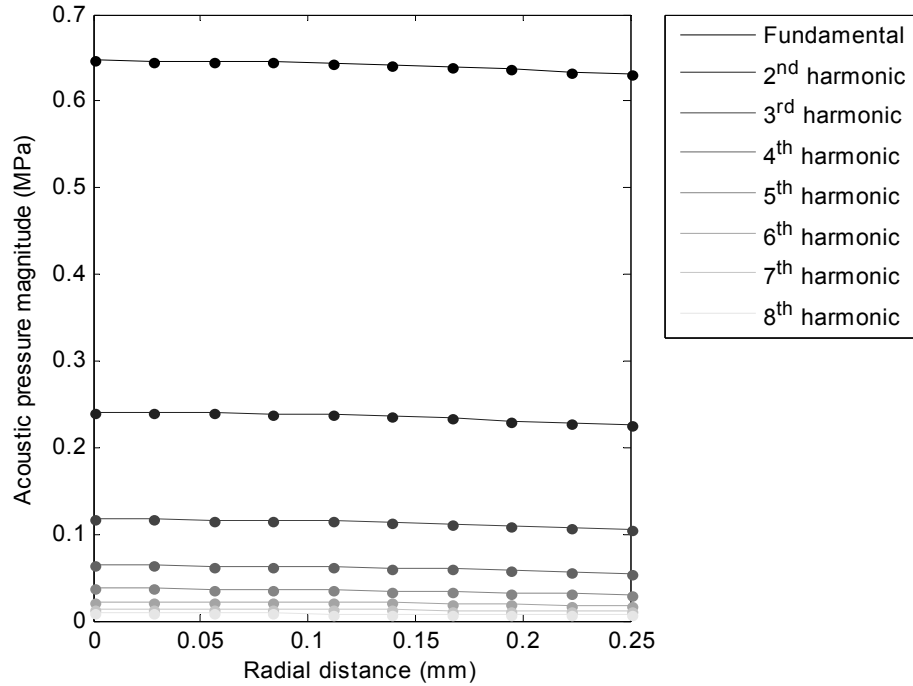


Fig. 25 – Radial variation of the pressure magnitude for first 8 harmonics at axial distance 60 mm from source, based on Bergen code field predictions. •: Radial positions at which cubic spline interpolation is used.

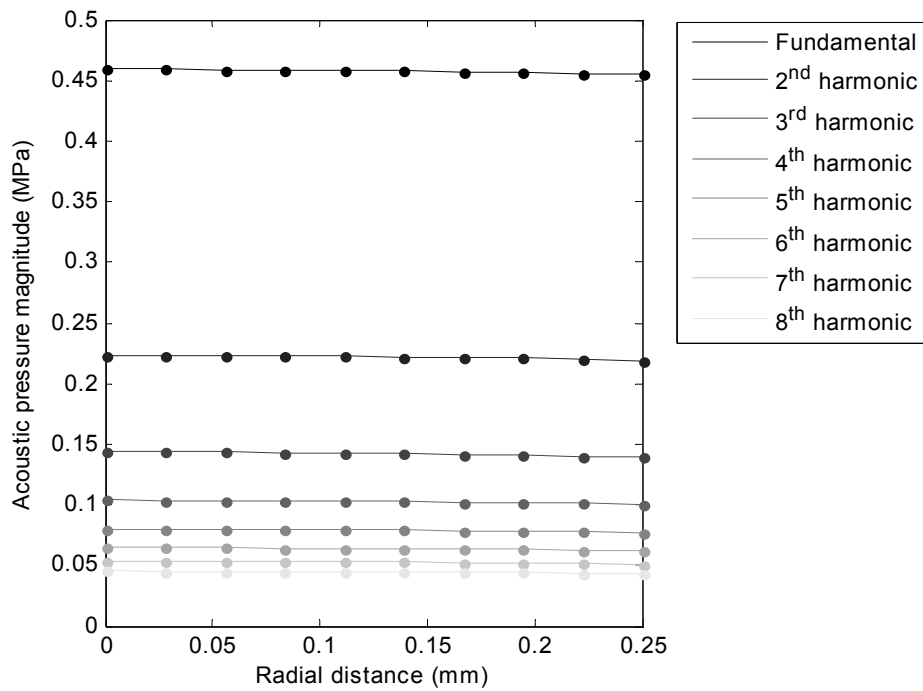


Fig. 26 – Radial variation of the pressure magnitude for first 8 harmonics at axial distance 90 mm from source, based on Bergen code field predictions. •: Radial positions at which cubic spline interpolation is used.

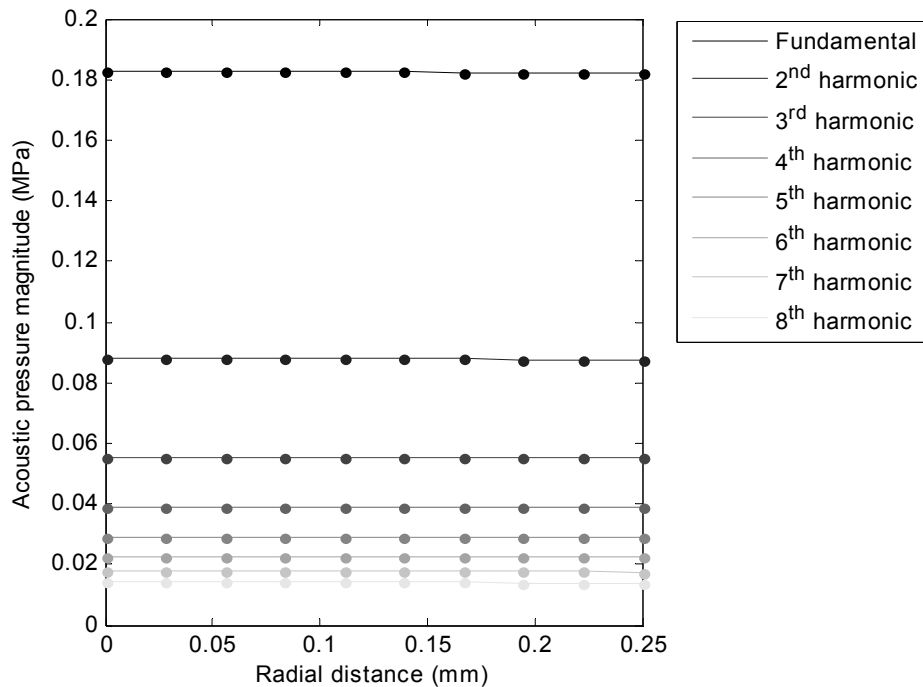


Fig. 27 – Radial variation of the pressure magnitude for first 8 harmonics at axial distance 180 mm from source, based on Bergen code field predictions. •: Radial positions at which cubic spline interpolation is used.

The output voltage of the hydrophone/amplifier system at the selected measurement positions, together with the ultrasonic pressure field data obtained from Bergen code

predictions, were then employed to estimate the loaded complex transfer characteristics of IP904, between 5 MHz and 100 MHz in 5 MHz steps, using equation (8). The loading correction effect of the Onda® AH-2010-100 amplifier was then removed using equation (3). An estimate of the complex open-circuit transfer characteristics was obtained at the following transducer-hydrophone separations:

- 90 mm
- 100 mm
- 110 mm
- 120 mm
- 140 mm
- 160 mm
- 180 mm.

At nearer separations, harmonics above 30 MHz have not yet reached their last axial maximum, hence introducing further uncertainty at these frequencies (see Fig. 24). Fig. 28 shows that the 100 MHz component reaches its last axial maximum at about 88.4 mm away from the source. It was therefore chosen not to carry out calibrations at separations nearer than 90 mm.

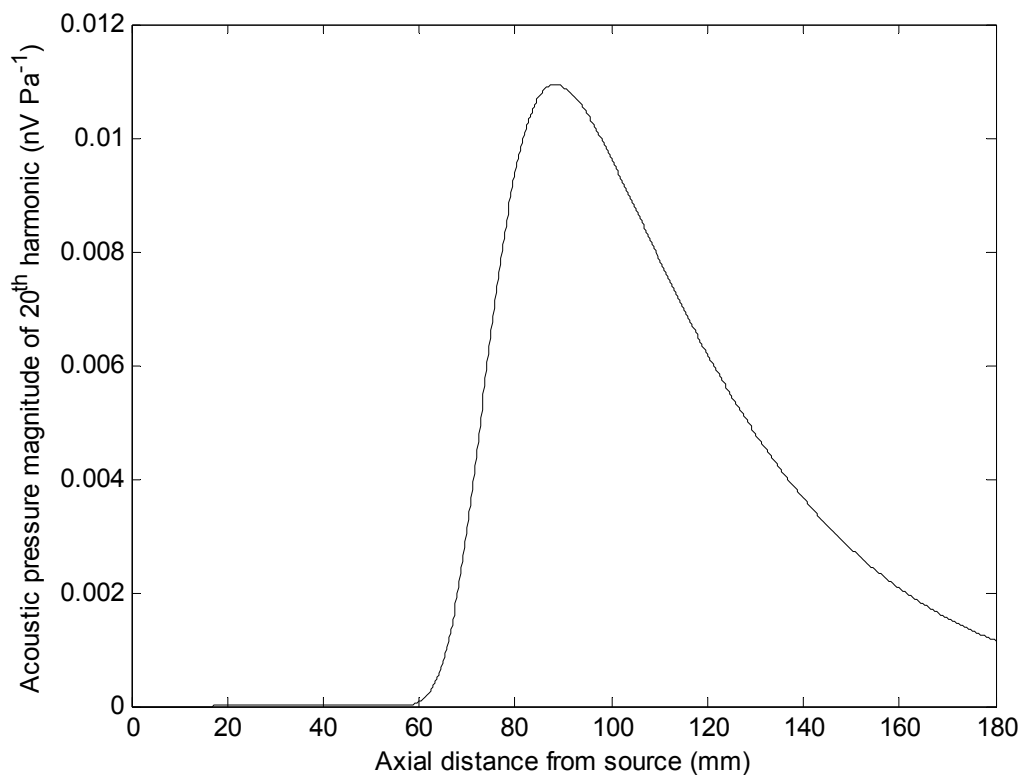


Fig. 28 – Bergen code field predictions with source parameters optimised to fit scanning tank measurement data. 100 MHz component.

Using the procedure outlined by Cooling and Humphrey (2007), the phases obtained at each axial position were expressed so as to have a common reference frequency. At each position z_i , a pure delay τ_i is obtained so that the time-shifted phase is zero at the

fundamental. Results for the magnitude and time-shifted relative phase transfer characteristics are shown in Figs. 29 and 30, respectively.

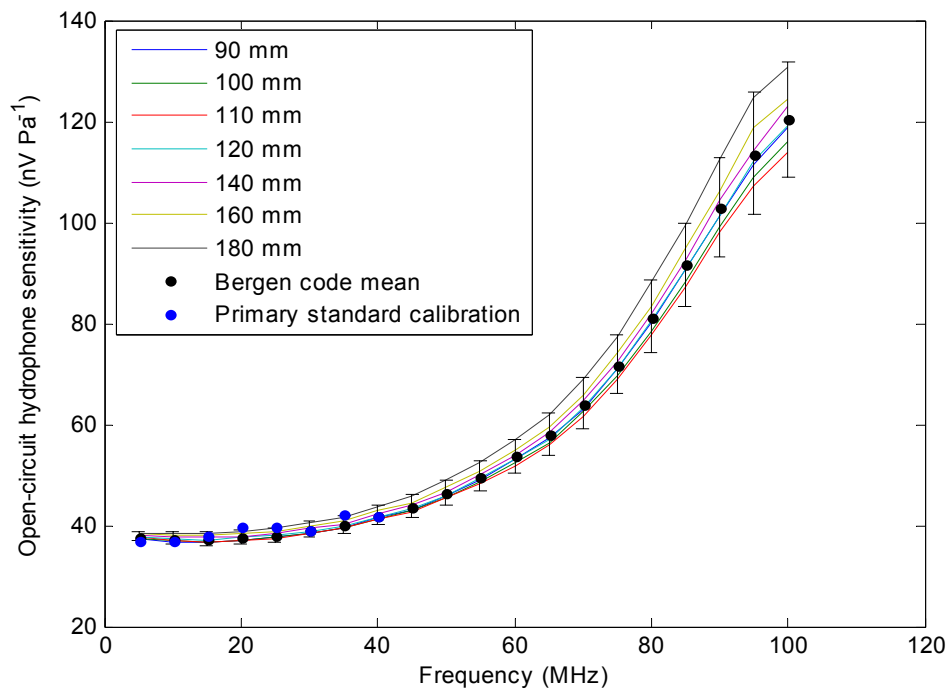


Fig. 29 – Sensitivity of IP904 estimated using equation (7) at transducer-hydrophone separations ranging between 90 mm and 180 mm. The mean value is overlaid, together with error bars corresponding to $\pm$ twice the standard deviation. The primary calibration values obtained in Section 3.2 are also overlaid.

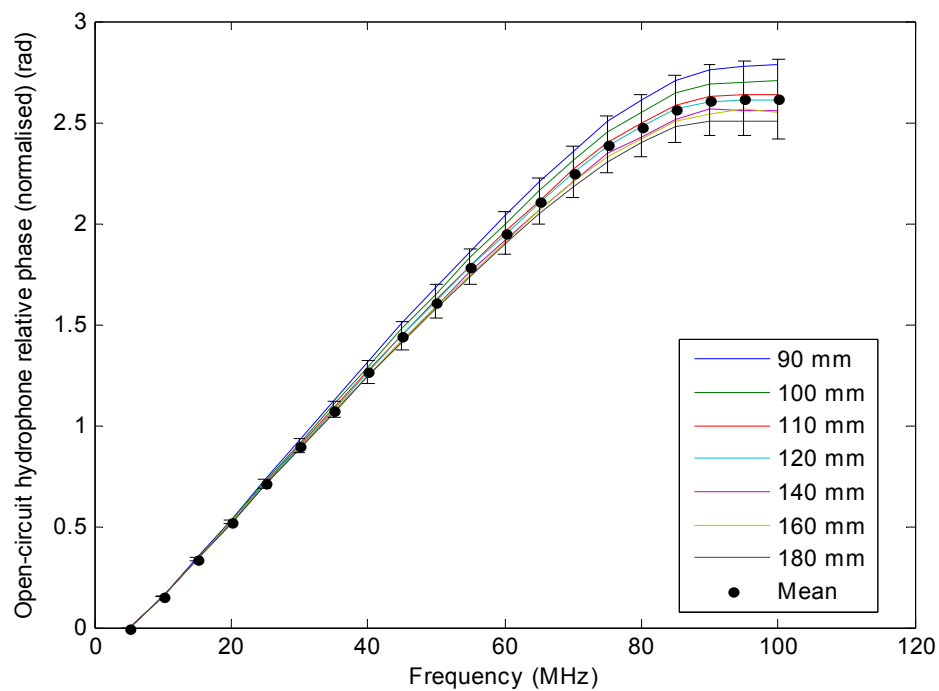


Fig. 30 – Time-shifted relative phase of IP904 estimated using equation (7) at transducer-hydrophone separations ranging between 90 mm and 180 mm. The time shift

process results in zero-phase at the fundamental. The mean value is overlaid, together with error bars corresponding to $\pm$ twice the standard deviation.

3.7 CHECKSOURCE WAVEFORM ACQUISITION AND DECONVOLUTION

After characterising the magnitude and phase response of the reference broadband hydrophone, it was of interest to test the deconvolution approach by estimating the true pressure waveform from a voltage measurement, and comparing this with results from optical methods.

NPL 3.5 MHz checksource CS001 operated on the high power setting was used to generate the waveforms. Acoustic pressure measurements were carried out at two separations, 51.5 mm and 80 mm away from the source, using IP904 plugged into a 70 MHz hydrophone amplifier (ref. 55643215). The amplifier was connected to a Tektronix® TDS784D oscilloscope via a 50 Ω shunt resistor. 100 single shot waveforms were acquired, and averaged. This procedure was repeated two times and an average waveform was calculated at each position.

Through knowledge of the hydrophone transfer characteristics displayed in Figs. 29 and 29, and by knowledge of the amplifier's complex gain and input impedance, the acoustic pressure waveform $p(t)$ at the hydrophone positions was estimated using the following equation:

$$p(t) = F^{-1} \left[\frac{V_{oc}(\omega)}{H_{oc}(\omega)} \right] \quad (9)$$

where

- $V_{oc}(\omega)$ is the open circuit hydrophone output voltage obtained from measurement
- $H_{oc}(\omega)$ is the open circuit hydrophone transfer characteristics displayed in Figs 29-30
- F^{-1} denotes the inverse Fourier transform.

The open circuit voltage is obtained from the loaded voltage by dividing by the loading factor $T(\omega)$ described in equation (3). The hydrophone transfer characteristics at frequencies which are not integer harmonics of 5 MHz are obtained using cubic spline interpolation, extrapolation being used between D.C. and 5 MHz.

Using the same procedure as in Section 3.2, the checksource waveform was measured using laser interferometry at 51.5 mm and 80 mm away from the source.

The time domain pressure waveforms obtained using equation (9) and via laser interferometry are shown in Figs. 31 and 32 and their FFT's in Figs. 33 and 34. It should be noted that some FIR filtering was applied to remove high frequency noise on the interferometer waveforms. The filter was of 200th order with a cut-off frequency of 90 MHz. Filtering was also applied to the deconvolved pressures to ensure consistency.

Estimates of the true pressure waveform were also obtained using the single value of the sensitivity at the fundamental (3.5 MHz) according to the procedure described in IEC 61157 (1992). These results are also displayed in Figs. 31-32.

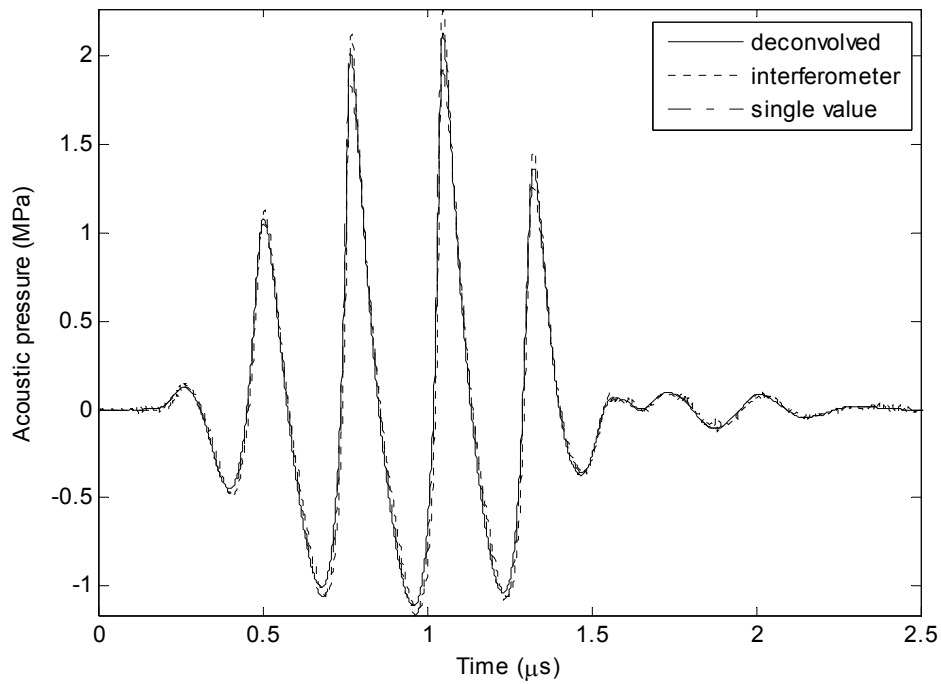


Fig. 31 – Acoustic pressure waveform derived from the electrical output waveform from checksource CS001 at 51.5 mm transducer/hydrophone separation. Comparison of waveform obtained through deconvolution with that obtained using laser interferometry and using a single value for the sensitivity at the fundamental.

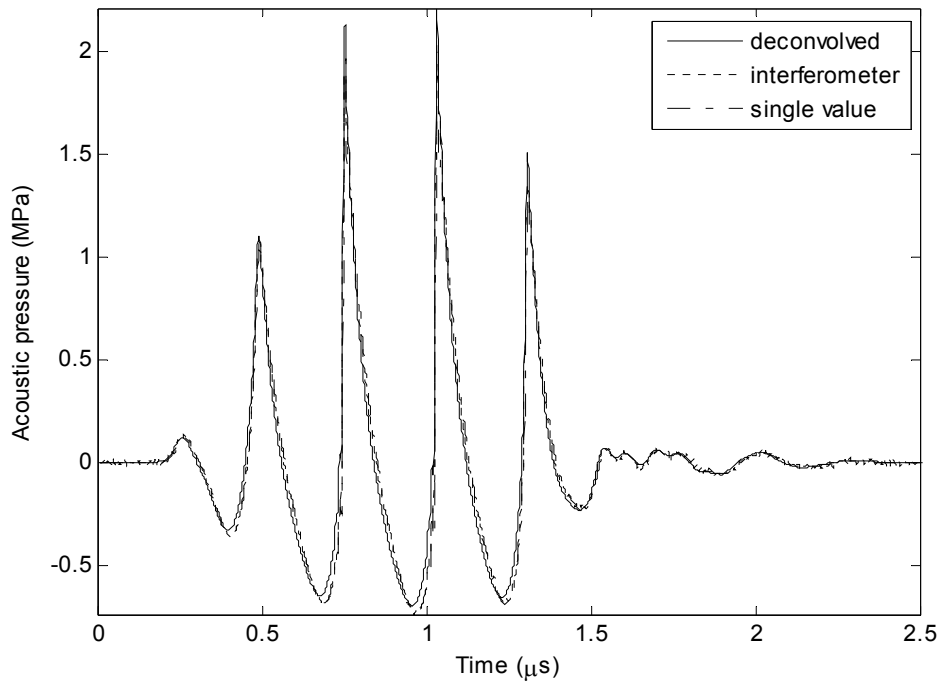


Fig. 32 – Acoustic pressure waveform derived from the electrical output waveform from checksource CS001 at 80 mm transducer/hydrophone separation. Comparison of waveform obtained through deconvolution with that obtained using laser interferometry and using a single value for the sensitivity at the fundamental.

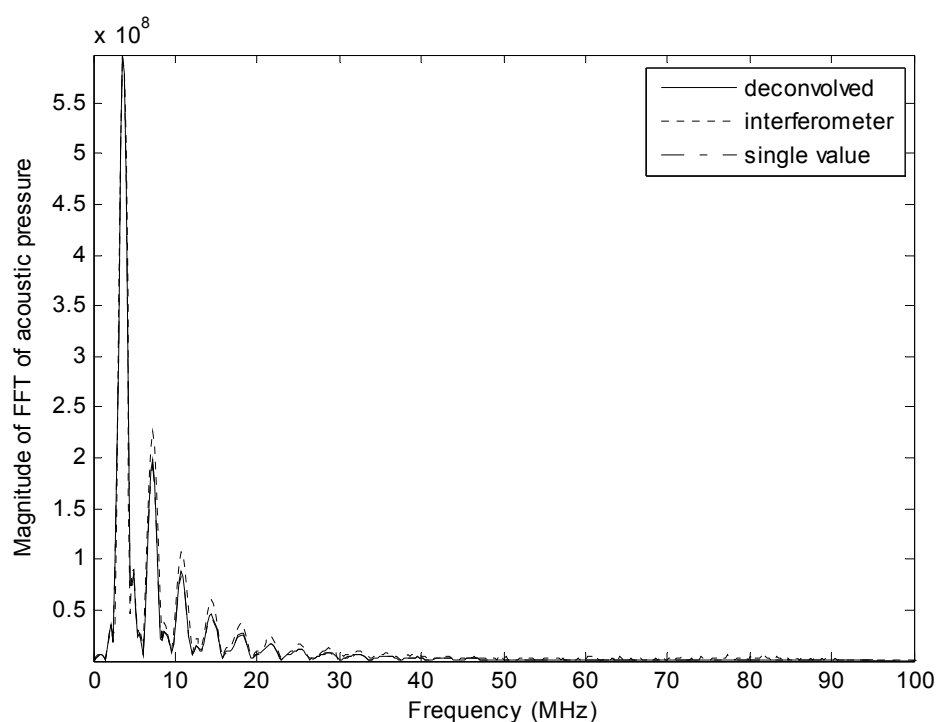


Fig. 33 – Magnitude of FFT of acoustic pressure waveform produced by checksource CS001 at 51.5 mm transducer/hydrophone separation. Comparison of FFT obtained through deconvolution with that obtained using laser interferometry and using a single value for the sensitivity at the fundamental.

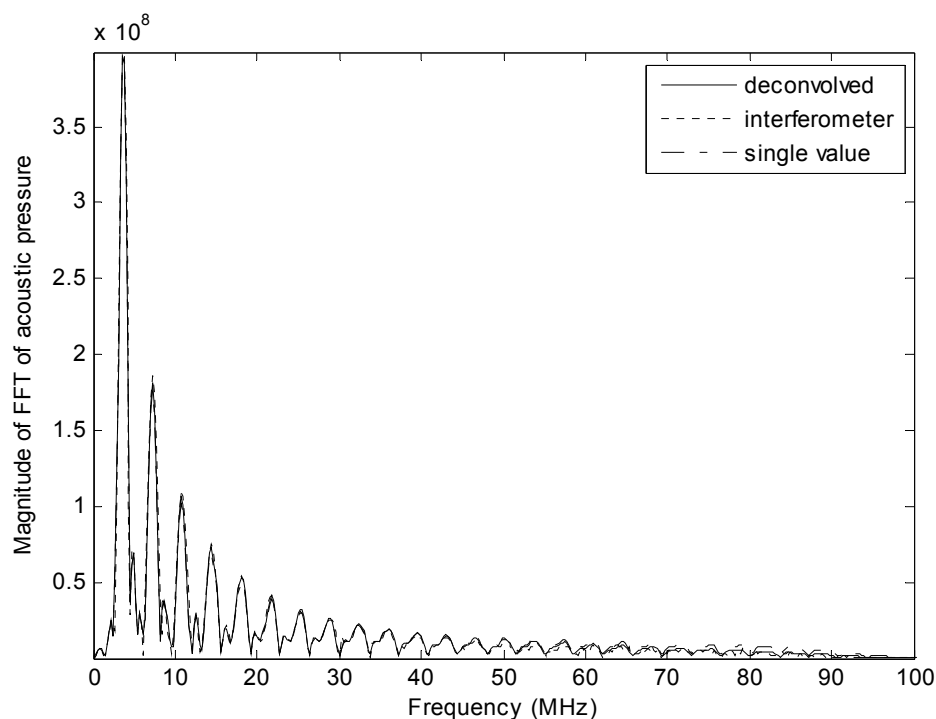


Fig. 34 – Magnitude of FFT of acoustic pressure waveform produced by checksource CS001 at 80 mm transducer/hydrophone separation. Comparison of FFT obtained through deconvolution with that obtained using laser interferometry and using a single value for the sensitivity at the fundamental.

From the data used to plot Figs. 31 and 32, the following pressure parameters, as defined by Preston (1991) are obtained for both the interferometer and the deconvolved waveforms.

- Peak-positive acoustic pressure, p_+ .
- Peak-negative acoustic pressure, p_- .
- Spatial-peak temporal-average intensity, I_{spta} .
- Pulse duration, t_d .

Results are shown in Tables 1 and 2.

Pressure parameter	Interferometer	Deconvolved	Percentage Error (Int.-Dec.)	Deconvolved (single value)	Percentage Error (Int.-Dec. sing. val.)
p_+ (MPa)	2.27	2.33	-2.9%	2.07	8.6%
p_- (MPa)	1.14	1.10	3.0%	1.16	-1.7%
I_{spta} (mW cm ⁻²)	97.2	97.3	-0.13%	96.3	0.95%
t_d (μs)	0.870	0.876	-0.65%	0.881	-1.2%

Table 1 – Comparison of pressure parameters between interferometer and deconvolved waveform at 51.5 mm from transducer.

Pressure parameter	Interferometer	Deconvolved	Percentage Error (Int.-Dec.)	Deconvolved (single value)	Percentage Error (Int.-Dec. sing. val.)
p_+ (MPa)	1.94	2.21	-14%	2.06	-6.2%
p_- (MPa)	0.737	0.697	5.5%	0.734	0.49%
I_{spta} (mW cm ⁻²)	50.8	50.9	0.10%	50.1	1.4%
t_d (μs)	0.987	0.986	0.13%	0.982	0.51%

Table 2 – Comparison of pressure parameters between interferometer and deconvolved waveform at 80 mm from transducer.

4. DISCUSSION

The methodology adopted in this study relies on an axisymmetric assumption for the ultrasonic source and pressure field. Clearly, from the raster scans shown in Section 3.1, the source does not behave in an axisymmetric way. Also, the assumption of piston-like movement of the transducer has not been entirely verified. This would require laser vibrometer scans across its face. Given that it is essentially the acoustic pressure close to the transducer axis that is of interest, it is likely that some deviation from axisymmetric behaviour is not problematic. This is reflected in how well the acoustic pressure field generated from the idealised source matches the measured field in Figs. 23 and 24. The non-uniformity of the piston velocity magnitude as a function of the radial coordinate could fairly easily be investigated using the cylindrical coordinate version of the Bergen code. Using laser vibrometer scans of a complete scan of the transducer face as input data to a three-dimensional implementation of the Bergen code would however present more challenges and may necessitate the use of distributed computing if any fit to measured data through optimisation is to be carried out. Generally, the question of how to quantify how well a transducer matches the behaviour of an ideal curved piston is raised, and how this relates to the uncertainty associated with the eventual calibration. Some thoughts on how this could be achieved are mentioned in Section 6.

The KZK equation relies on the use of the linear plane-progressive-wave impedance relation to convert between pressure and particle velocity. It turns out that this assumption is suitable for this study, as for a focused source, the sound beam is directional and is localised in the vicinity of the transducer axis. Nevertheless, its current implementation relies on finding the values of u_0 , a , D , β and δ for which the predicted field best matches the measured field over the first 8 harmonics. Although the fit is a good match to measured data up to 40 MHz, there is less confidence that pressure field predictions up to 100 MHz will be representative of what is happening in the propagating medium. Nevertheless, the predicted magnitude response of the broadband membrane hydrophone is in accordance with what would be expected of such a device, its sensitivity increasing as a function of frequency. The first resonance of a 9 μm film thickness coplanar membrane hydrophone is known to be in excess of 90 MHz (Robinson 1991). The fact that resonant behaviour is not clearly predicted here could be because the *pvd*f film thickness is less than the quoted value, as a result of the manufacturing process. According to the hydrophone model described by G  lat et al (2005), a 7 μm film *pvd*f thickness coplanar device would have a first resonant frequency at 108 MHz.

When comparing the hydrophone output voltage with the predicted axial pressure, averaged over the hydrophone receiving area, it is important to select axial positions where sufficient high frequency content exists. Although the focal distance for the transducer used is quoted as 60 mm, this figure relates to the fundamental frequency under low-drive conditions (see Fig. 3). Under the drive conditions investigated here, it is after 80 mm that the 100 MHz component reaches its last axial maximum. Although the magnitude and phase values obtained at axial positions between 90 mm and 180 mm are fairly consistent, the spread increases with frequency, with a standard deviation of 1.1% of the mean at 5 MHz and 4.7% of the mean at 100 MHz for the magnitude. Calculating these figures for the relative phase response of the hydrophone is not quite

as meaningful, since the shape and gradient of the phase curve is highly sensitive to the value of the pure delay by which the true phase response has been shifted. For example, if 10 MHz were chosen as the frequency at which the relative phase response were to be zero, the standard deviation for the relative phase at 100 MHz is 0.22 radians, whereas this quantity is 0.098 radians when 5 MHz is chosen as the zero phase frequency.

It should be noted that the waveforms acquired are subject to usual uncertainties: hydrophone alignment, movement of hydrophone membrane, variations in water temperature, etc. Jitter is minimised by alignment of waveforms at the post-processing stage. Although 100 waveforms were acquired at each measurement position, it is possible that more acquisitions may further reduce noise at higher order harmonics. As the magnitude of a harmonic decreases as the order of the harmonic increases, at the measurement positions where calibrations were carried out, the signal-to-noise ratio will decrease accordingly. It may therefore be advantageous to increase the number of averages. It is however possible that acquisition times may be significantly increased and a compromise may have to be sought.

Another source of uncertainty includes the KZK equation itself. Aside from issues arising from simplification of the equation due to the parabolic approximation, it is not clear whether this equation has been validated at the frequencies used as part of this feasibility study.

Interpolation, as well as extrapolation to obtain hydrophone frequency response at inter-harmonic frequencies and frequencies between D.C. and the fundamental, respectively, will also yield further source of uncertainty. Given that the response of a membrane device is both broadband and smooth (Robinson 1991), interpolation should not cause too many problems. Extrapolation however results in potential bigger sources of uncertainty, particularly if the hydrophone transfer characteristics are used when the fundamental frequency is below 5 MHz, as in Section 3.7. If a needle hydrophone were to be characterised using this methodology, it is likely that issues arising from interpolation and extrapolation will become more significant, due to variations in the frequency response that are present in these devices. There also is some debate about the importance of obtaining low-frequency information, even if the frequency content of the signals under investigation is within the range where the hydrophone has been characterised. It is speculated by Harris et al (1995) that the mechanical index is affected by lack of knowledge of low-frequency hydrophone data.

It should be noted that effects due to spatial averaging due to finite hydrophone element receiving area tend to become smaller as the transducer-hydrophone separation increases beyond the last axial maximum and the beam gets wider: at the fundamental, the change in the pressure magnitude between $r = 0$ and $r = 0.25$ mm is 2.4 % at $z = 60$ mm. At $z = 180$ mm, this figure is reduced to 0.24 %. Also, effects due to spatial averaging tend to increase with frequency. At $z = 180$ mm, the pressure magnitude of the 100 MHz harmonic changes by 2.0 % between $r = 0$ and $r = 0.25$ mm.

Evaluation of the pressure parameters in Tables 1 and 2 shows that the peak-positive pressure estimated through waveform deconvolution using the hydrophone transfer characteristics obtained as part of this study agrees within 3% with interferometer data at a transducer-hydrophone separation of 51.5 mm, but shows a +14% difference with interferometer measurement at 80 mm. This is likely to be due to the fact that the

frequency content of the waveform acquired using laser interferometry is noisy beyond 50 MHz (see Fig. 34), due to current limitations of the laser interferometer used. At 80 mm, significant frequency content above 50 MHz exists, which will contribute to the observed percentage difference in p_+ . Using single frequency deconvolution, i.e. scaling the measurement system output voltage waveform by the hydrophone/amplifier sensitivity at the fundamental frequency (3.5 MHz), the percentage difference in p_+ compared with results obtained by deconvolving the complete transfer characteristics shows that this quantity is underestimated by 11% at 51.5 mm separation and by 7% at 80 mm (see Tables 1 and 2). This demonstrates that even for a relatively broadband measurement, knowledge of the complete transfer characteristics may be required if an accurate estimate of p_+ is needed. It should be noted that these difficulties in determining p_+ are known, and have contributed to why it is not required to be declared by specification standards (IEC 61157 1992).

Other quantities in Tables 1 and 2 are estimated within $\pm 6\%$. It is worth noting that current laser interferometry capabilities at NPL will have to be improved upon in order to fully assess the calibration procedure resulting from this feasibility study. Better measurement capability beyond 50 MHz is required.

5. CONCLUSION

A feasibility study aimed at assessing and improving a hydrophone calibration method employing a combination of ultrasonic field measurement and nonlinear field prediction has been carried out. The magnitude and relative phase characteristics of a broadband membrane hydrophone were obtained between 5 MHz and 100 MHz by comparing its open-circuit output voltage with the theoretically predicted ultrasonic pressure at the hydrophone location in a harmonically rich field. The methodology employed relies on an axisymmetric implementation of the KZK equation, but differs from previous work (Cooling and Humphrey 2007, Bleeker and Lewin 2000) in that prior knowledge of the hydrophone sensitivity between 5 MHz and 40 MHz is assumed and that the predicted field is optimised against the measured field over this frequency range by varying the source and medium parameters. The predicted hydrophone magnitude response shows consistency within $\pm 10\%$ at transducer-hydrophone separations between 90 mm and 180 mm.

The radial distribution of the ultrasonic field has been mapped over the radius of the hydrophone active element at selected axial positions, hence providing further insight into effects of spatial averaging due to finite hydrophone element receiving area.

3.5 MHz fundamental frequency checksource waveforms were acquired using both the broadband membrane hydrophone employed in this study and a laser interferometer. At an axial position (51.5 mm) where the frequency content of the acoustic pressure is within the bandwidth of the interferometer, agreement for p_+ and p_- is within 3% and better than 1% for I_{spta} and t_d . At a transducer-hydrophone separation (80 mm) where the frequency content of the acoustic pressure goes beyond the bandwidth of the interferometer, p_+ estimated using optical methods appears 14% lower than in the deconvolved pressure waveform. This highlights the need for better measurement capability of laser interferometry beyond 50 MHz.

Despite the underlying assumptions of the KZK equation, as well as the simple model adopted for the source, this method shows promise in:

- assisting the development of laser interferometry beyond 50 MHz;
- aiding field mapping of transducers in the axial and radial directions at field positions where the KZK equations yield reliable results (see Equations 4 and 5);
- providing magnitude and phase calibration of other hydrophones by knowledge of the transfer characteristics of the broadband hydrophone IP904 described in this feasibility study, i.e. by using it as the reference device.

6. FURTHER WORK

This feasibility study has yielded useful results and validated previous work (Cooling and Humphrey 2007, Bleeker and Lewin 2000), and also points towards a number of areas that would benefit from further research.

Although a curved rigidly vibrating circular piston appears to generate a field that agrees well with measurement up to 40 MHz, it is currently not known how well this matches the transducer displacement profile on its front face. Raster scans will produce a measurement of the acoustic pressures close to the face of the transducer, giving some indication of how the transducer behaves. Producing surface scans using a laser vibrometer will give a better indication of how the transducer behaviour differs from that of a plane concave piston, and it would even be possible to use this data as input to the Bergen code. Due to the curved nature of the transducer front face, it is likely that this will present some challenges. Other means of simulating the source in a more realistic way may also be investigated, such as imposing a windowing function on the piston source profile or using finite element modelling. Investigating the field harmonic content beyond 40 MHz after the last axial maxima and comparing with the plane piston case would then provide a better idea of how adequate the current concave piston model is.

Given that the KZK equation relies on specific assumptions described in Section 2.4.1, it may be worth exploring other options for field modelling, such as the Westervelt equation, or a finite volume approach, which looks at the fundamentals of nonlinear wave propagation by considering the conservation of energy inside an infinitesimal volume of fluid.

To complement this work, it is vital that better laser interferometer measurements be carried out, with improved signal to noise ratio at frequencies above 40 MHz. This will provide further validation of the methodology outlined in this paper and conversely, the procedure described in this feasibility study may help provide confidence in calibrations obtained using laser interferometry.

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ANNEX A – THEORY

The software kzkcy1_07.exe solves the equations

$$\begin{aligned}\frac{\partial g_n}{\partial \sigma} &= \frac{np_{nonlin}}{2(1+\sigma)} \left\{ \sum_{j=1}^{n-1} h_j g_{n-j} + \sum_{j=n+1}^N (g_{j-n} h_j - h_{j-n} g_j) \right\} - n^2 p_{abs} g_n - \frac{\nabla_X^2 h_n}{4n(1+\sigma)^2}, \\ \frac{\partial h_n}{\partial \sigma} &= \frac{np_{nonli}}{2(1+\sigma)} \left\{ \frac{1}{2} \sum_{j=1}^{n-1} (h_j h_{n-j} - g_j g_{n-j}) - \sum_{j=n+1}^N (g_j g_{j-n} + h_j h_{j-n}) \right\} - n^2 p_{abs} h_n + \frac{\nabla_X^2 g_n}{4n(1+\sigma)^2}, \\ n &= 1, 2, \dots, N\end{aligned}$$

where ∇_X^2 denotes the radial term in the Laplacian operator.

These equations are derived from setting

$$\begin{aligned}P &= \sum_{n=1}^N [g_n(X, \sigma) \cos(nT) + h_n(X, \sigma) \sin(nT)], \\ &= \frac{1}{2} \left[\sum_{n=1}^N P_n(X, \sigma) e^{-inT} + \sum_{n=1}^N P_n^*(X, \sigma) e^{inT} \right] \\ &= \text{Real} \left[\sum_{n=1}^N P_n(X, \sigma) e^{-inT} \right]\end{aligned}$$

where P^* is the complex conjugate of P , in the equation

$$\frac{\partial P}{\partial \sigma} = \frac{p_{nonlin}}{(1+\sigma)} P \frac{\partial P}{\partial T} + p_{abs} \frac{\partial^2 P}{\partial T^2} + \frac{1}{4(1+\sigma)^2} \int_{-\infty}^T \nabla_X^2 P(X, \sigma, \hat{T}) d\hat{T}.$$

This last equation can be derived from the KZK equation

$$\frac{\partial p}{\partial z} = \frac{\beta p}{\rho_0 c_0^3} \frac{\partial p}{\partial \tau} + \frac{\delta}{2c_0^3} \frac{\partial^2 p}{\partial \tau^2} + \frac{c_0}{2} \int_{-\infty}^{\tau} \nabla_{\perp}^2 p(r, z, \hat{\tau}) d\hat{\tau}$$

by using the substitutions

$$\begin{aligned}\sigma &= \frac{z}{z_0}, \\ X &= \frac{r/a}{1+z/z_0}, \\ T &= \omega_0 \tau - \frac{r^2/a^2}{1+z/z_0}, \\ P &= \frac{p(r, z, \tau)(1+z/z_0)}{P_0},\end{aligned}$$

where $z_0 = \omega_0 a^2 / 2c_0$. Using these substitutions leads to

$$p_{ab} = \frac{\beta p_0 \omega_0^2 a^2}{2 \rho_0 c_0^4},$$

$$p_{nonlin} = \frac{\delta a^2 \omega_0^3}{4 c_0^4}.$$

The software outputs values of $p_0 g_n / (1 + \sigma)$ and $p_0 h_n / (1 + \sigma)$ evaluated at $r = 0$ (and hence $X = 0$) to the file R0HARMS.TXT, labelled as real and imaginary parts respectively (because they are the real and imaginary parts of the n^{th} harmonic of p).

Details of the conversion of one equation to another

This section covers the details of going from the KZK equation to the transformed version, and from the transformed version to the equations for individual components.

Transforming the KZK equation

Consider a function $p(r, z, \tau)$ that obeys the KZK equation, and consider the effects of the first three coordinate transforms on the terms in the KZK equation:

$$\int_{-\infty}^{\tau} p(r, z, \hat{\tau}) d\hat{\tau} = \frac{1}{\omega_0} \int_{-\infty}^T p(r, z, \hat{T}) d\hat{T},$$

$$\frac{\partial p}{\partial \tau} = \frac{\partial p}{\partial T} \frac{\partial T}{\partial \tau} = \omega_0 \frac{\partial p}{\partial T},$$

$$\frac{\partial^2 p}{\partial \tau^2} = \omega_0^2 \frac{\partial^2 p}{\partial T^2},$$

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial X} \frac{\partial X}{\partial r} + \frac{\partial p}{\partial T} \frac{\partial T}{\partial r},$$

$$= \frac{\partial p}{\partial X} \frac{1/a}{(1 + z/z_0)} - \frac{\partial p}{\partial T} \frac{2r/a^2}{(1 + z/z_0)},$$

$$\frac{\partial^2 p}{\partial r^2} = \frac{\partial^2 p}{\partial X^2} \frac{1/a^2}{(1 + z/z_0)^2} - \frac{\partial^2 p}{\partial T \partial X} \frac{4r/a^3}{(1 + z/z_0)^2} + \frac{\partial^2 p}{\partial T^2} \frac{4r^2/a^4}{(1 + z/z_0)^2} - \frac{\partial p}{\partial T} \frac{2/a^2}{(1 + z/z_0)},$$

$$\frac{\partial p}{\partial z} = \frac{\partial p}{\partial X} \frac{\partial X}{\partial z} + \frac{\partial p}{\partial T} \frac{\partial T}{\partial z} + \frac{\partial p}{\partial \sigma} \frac{\partial \sigma}{\partial z},$$

$$= -\frac{\partial p}{\partial X} \frac{r/a}{z_0(1 + z/z_0)^2} + \frac{\partial p}{\partial T} \frac{r^2/a^2}{z_0(1 + z/z_0)^2} + \frac{1}{z_0} \frac{\partial p}{\partial \sigma}.$$

Hence

$$\begin{aligned}
\nabla_r^2 p &= \frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r}, \\
&= \frac{\partial^2 p}{\partial X^2} \frac{1/a^2}{(1+z/z_0)^2} - \frac{\partial^2 p}{\partial T \partial X} \frac{4r/a^3}{(1+z/z_0)^2} + \frac{\partial^2 p}{\partial T^2} \frac{4r^2/a^4}{(1+z/z_0)^2} - \frac{\partial p}{\partial T} \frac{2/a^2}{(1+z/z_0)} \\
&\quad + \frac{1}{r} \left(\frac{\partial p}{\partial X} \frac{1/a}{(1+z/z_0)} - \frac{\partial p}{\partial T} \frac{2r/a^2}{(1+z/z_0)} \right), \\
&= \frac{1/a^2}{(1+z/z_0)^2} \left(\frac{\partial^2 p}{\partial X^2} + \frac{1}{X} \frac{\partial p}{\partial X} \right) + \frac{4}{a^2} \left[\frac{\partial^2 p}{\partial T^2} \frac{r^2/a^2}{(1+z/z_0)^2} - \frac{\partial^2 p}{\partial T \partial X} \frac{r/a}{(1+z/z_0)^2} - \frac{\partial p}{\partial T} \frac{1}{(1+z/z_0)} \right] \\
&= \left(\frac{1/a}{1+z/z_0} \right)^2 \nabla_X^2 p + \frac{4/a^2}{(1+z/z_0)^2} \left[\frac{r^2}{a^2} \frac{\partial^2 p}{\partial T^2} - \frac{r}{a} \frac{\partial^2 p}{\partial T \partial X} - (1+z/z_0) \frac{\partial p}{\partial T} \right]
\end{aligned}$$

and so

$$\int_{-\infty}^{\tau} \nabla_r^2 p d\hat{\tau} = \frac{1}{\omega_0} \left(\frac{1/a}{1+z/z_0} \right)^2 \int_{-\infty}^{\tau} \nabla_X^2 p d\hat{T} + \frac{4/a^2}{\omega_0} \left[\frac{r^2/a^2}{(1+z/z_0)^2} \frac{\partial p}{\partial T} - \frac{r/a}{(1+z/z_0)^2} \frac{\partial p}{\partial X} - \frac{1}{(1+z/z_0)} p \right].$$

Inserting all terms into the KZK equation gives

$$\begin{aligned}
\frac{1}{z_0} \left[\frac{\partial p}{\partial \sigma} - \frac{\partial p}{\partial X} \frac{r/a}{(1+z/z_0)^2} + \frac{\partial p}{\partial T} \frac{r^2/a^2}{(1+z/z_0)^2} \right] &= \frac{\beta \omega_0}{\rho_0 c_0^3} p \frac{\partial p}{\partial T} + \frac{\delta \omega_0^2}{2c_0^3} \frac{\partial^2 p}{\partial T^2} \\
+ \frac{c_0}{2\omega_0} \left(\frac{1/a}{1+z/z_0} \right)^2 \int_{-\infty}^{\tau} \nabla_X^2 p d\hat{T} &+ \frac{2c_0}{a^2 \omega_0} \left[\frac{\partial p}{\partial T} \frac{r^2/a^2}{(1+z/z_0)^2} - \frac{\partial p}{\partial X} \frac{r/a}{(1+z/z_0)^2} - p \frac{1}{(1+z/z_0)} \right]
\end{aligned}$$

Hence setting $z_0 = \omega_0 a^2 / 2c_0$ and multiplying by z_0 eliminates several derivative terms, leaving

$$\begin{aligned}
\frac{\partial p}{\partial \sigma} &= \frac{\beta \omega_0^2 a^2}{2\rho_0 c_0^4} p \frac{\partial p}{\partial T} + \frac{\delta \omega_0^3 a^2}{4c_0^4} \frac{\partial^2 p}{\partial T^2} - \frac{p}{(1+z/z_0)} + \frac{1}{4(1+z/z_0)^2} \int_{-\infty}^{\tau} \nabla_X^2 p d\hat{T} \\
&= \frac{\beta \omega_0^2 a^2}{2\rho_0 c_0^4} p \frac{\partial p}{\partial T} + \frac{\delta \omega_0^3 a^2}{4c_0^4} \frac{\partial^2 p}{\partial T^2} - \frac{p}{(1+\sigma)} + \frac{1}{4(1+\sigma)^2} \int_{-\infty}^{\tau} \nabla_X^2 p d\hat{T}
\end{aligned}$$

Now consider the derivatives of the expression for p .

$$\begin{aligned}
\frac{\partial p}{\partial X} &= \frac{p_0}{(1+\sigma)} \frac{\partial P}{\partial X}, \\
\frac{\partial p}{\partial T} &= \frac{p_0}{(1+\sigma)} \frac{\partial P}{\partial T}, \\
\frac{\partial p}{\partial \sigma} &= \frac{p_0}{(1+\sigma)} \frac{\partial P}{\partial \sigma} - \frac{p_0 P}{(1+\sigma)^2}
\end{aligned}$$

and substituting these expressions into the partially transformed equation gives

$$\begin{aligned} \frac{p_0}{(1+\sigma)} \frac{\partial P}{\partial \sigma} - \frac{p_0 P}{(1+\sigma)^2} &= \frac{\beta \omega_0^2 a^2}{2\rho_0 c_0^4} \frac{p_0^2 P}{(1+\sigma)^2} \frac{\partial P}{\partial T} + \frac{\delta \omega_0^3 a^2}{4c_0^4} \frac{p_0}{(1+\sigma)} \frac{\partial^2 P}{\partial T^2} \\ &\quad - \frac{p_0 P}{(1+\sigma)^2} + \frac{p_0}{4(1+\sigma)^3} \int_{-\infty}^T \nabla_X^2 P d\hat{T} \end{aligned}$$

Eliminating terms and multiplying throughout by $(1+\sigma)/p_0$ leads to

$$\frac{\partial P}{\partial \sigma} = \frac{\beta \omega_0^2 a^2 p_0}{2\rho_0 c_0^4} \frac{P}{(1+\sigma)} \frac{\partial P}{\partial T} + \frac{\delta \omega_0^3 a^2}{4c_0^4} \frac{\partial^2 P}{\partial T^2} + \frac{1}{4(1+\sigma)^2} \int_{-\infty}^T \nabla_X^2 P d\hat{T}$$

and so setting

$$p_{ab} = \frac{\beta p_0 \omega_0^2 a^2}{2\rho_0 c_0^4},$$

$$p_{nonlin} = \frac{\delta a^2 \omega_0^3}{4c_0^4}.$$

gives

$$\frac{\partial P}{\partial \sigma} = \frac{p_{nonlin}}{(1+\sigma)} P \frac{\partial P}{\partial T} + p_{ab} \frac{\partial^2 P}{\partial T^2} + \frac{1}{4(1+\sigma)^2} \int_{-\infty}^T \nabla_X^2 P(X, \sigma, \hat{T}) d\hat{T}.$$

which is the modified form given in the first section above.

Substitution of the series solution

Assume an infinite series solution of the form

$$\begin{aligned} P &= \sum_{n=1}^{\infty} [g_n(X, \sigma) \cos(nT) + h_n(X, \sigma) \sin(nT)] \\ &= \frac{1}{2} \left[\sum_{n=1}^{\infty} P_n(X, \sigma) e^{-inT} + \sum_{n=1}^{\infty} P_n^*(X, \sigma) e^{inT} \right] \end{aligned}$$

Considering the required derivatives and integrals with respect to T (and neglecting explicit mention of dependence on X and σ):

$$\begin{aligned} \frac{\partial P}{\partial T} &= \frac{1}{2} \left[\sum_{n=1}^{\infty} -in (P_n e^{-inT} - P_n^* e^{inT}) \right] \\ \frac{\partial^2 P}{\partial T^2} &= \frac{1}{2} \left[\sum_{n=1}^{\infty} -n^2 (P_n e^{-inT} + P_n^* e^{inT}) \right] \\ \int_{-\infty}^T P d\hat{T} &= \frac{1}{2} \left[\sum_{n=1}^{\infty} \frac{i}{n} (P_n e^{-inT} - P_n^* e^{inT}) \right] \end{aligned}$$

The nonlinear term then becomes

$$\begin{aligned}
P \frac{\partial P}{\partial T} &= \frac{1}{4} \left[\sum_{m=1}^{\infty} (P_m e^{-imT} + P_m^* e^{imT}) \right] \left[\sum_{n=1}^{\infty} -in (P_n e^{-inT} - P_n^* e^{inT}) \right] \\
&= \frac{1}{4} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -in (P_m P_n e^{-i(m+n)T} - P_m^* P_n^* e^{i(m+n)T}) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -in (P_m^* P_n e^{-i(n-m)T} - P_m P_n^* e^{i(n-m)T}) \right\}.
\end{aligned}$$

Consider these two terms one at a time. Let $q = m+n$ in the first term.

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -in (P_m P_n e^{-i(m+n)T} - P_m^* P_n^* e^{i(m+n)T}) = \sum_{q=2}^{\infty} \sum_{n=1}^{q-1} -in (P_{q-n} P_n e^{-iqT} - P_{q-n}^* P_n^* e^{iqT}).$$

Consider the sum over n for odd $q=2k+1$ (so that $q-1=2k$ is even). Then

$$\begin{aligned}
&\sum_{n=1}^{q-1} -in (P_{q-n} P_n e^{-iqT} - P_{q-n}^* P_n^* e^{iqT}) = \sum_{n=1}^k -in (P_{q-n} P_n e^{-iqT} - P_{q-n}^* P_n^* e^{iqT}) + \sum_{n=k+1}^{2k} -in (P_{q-n} P_n e^{-iqT} - P_{q-n}^* P_n^* e^{iqT}) \\
&= \sum_{n=1}^k -in (P_{q-n} P_n e^{-iqT} - P_{q-n}^* P_n^* e^{iqT}) + \sum_{t=1}^k -i(q-t) (P_t P_{q-t} e^{-iqT} - P_t^* P_{q-t}^* e^{iqT}) \\
&= \sum_{n=1}^k -iq (P_{q-n} P_n e^{-iqT} - P_{q-n}^* P_n^* e^{iqT}) \\
&= \frac{1}{2} \sum_{n=1}^{2k} -iq (P_{q-n} P_n e^{-iqT} - P_{q-n}^* P_n^* e^{iqT})
\end{aligned}$$

where the second term was transformed using $t = q - n$ and was then combined with the first term. If q is even, $q=2k$, splitting the sum over n into two sums from 1 to $k-1$ and $k+1$ to $2k$ plus the term for $n = k$ and using the same methods as above shows that

$$\sum_{n=1}^{q-1} -in (P_{q-n} P_n e^{-iqT} - P_{q-n}^* P_n^* e^{iqT}) = \frac{1}{2} \sum_{n=1}^{q-1} -iq (P_{q-n} P_n e^{-iqT} - P_{q-n}^* P_n^* e^{iqT})$$

whatever the value of q . Now consider the other term in the equation for the nonlinear term and note that terms with $n=m$ will vanish, so the sum can be split into terms where $n > m$ and terms where $n < m$:

$$\begin{aligned}
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -in (P_m^* P_n e^{-i(n-m)T} - P_m P_n^* e^{i(n-m)T}) &= \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} -in (P_m^* P_n e^{-i(n-m)T} - P_m P_n^* e^{i(n-m)T}) \\
&\quad + \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} -in (P_m^* P_n e^{-i(n-m)T} - P_m P_n^* e^{i(n-m)T}) \\
&= \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} -in (P_m^* P_n e^{-i(n-m)T} - P_m P_n^* e^{i(n-m)T}) \\
&\quad - \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} -in (P_n^* P_m e^{-i(m-n)T} - P_n P_m^* e^{i(m-n)T})
\end{aligned}$$

Let $r=n-m$ in the first term and $s=m-n$ in the second term. Then

$$\begin{aligned}
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} -in \left(P_m^* P_n e^{-i(n-m)T} - P_m P_n^* e^{i(n-m)T} \right) &= \sum_{n=2}^{\infty} \sum_{r=1}^{n-1} -in \left(P_{n-r}^* P_n e^{-irT} - P_{n-r} P_n^* e^{irT} \right) \\
&\quad - \sum_{m=2}^{\infty} \sum_{s=1}^{m-1} -i(m-s) \left(P_{m-s}^* P_m e^{-isT} - P_{m-s} P_m^* e^{isT} \right) \\
&= \sum_{m=2}^{\infty} \sum_{s=1}^{m-1} -is \left(P_{m-s}^* P_m e^{-isT} - P_{m-s} P_m^* e^{isT} \right) \\
&= \sum_{s=1}^{\infty} \sum_{m=s+1}^{\infty} -is \left(P_{m-s}^* P_m e^{-isT} - P_{m-s} P_m^* e^{isT} \right)
\end{aligned}$$

Substituting these expressions into the nonlinear term gives

$$\begin{aligned}
P \frac{\partial P}{\partial T} &= \frac{-i}{4} \left(\frac{1}{2} \sum_{r=1}^{\infty} \sum_{n=1}^{r-1} r \left(P_{r-n} P_n e^{-irT} - P_{r-n}^* P_n^* e^{irT} \right) + \sum_{r=1}^{\infty} \sum_{n=r+1}^{\infty} r \left(P_{n-r}^* P_n e^{-irT} - P_{n-r} P_n^* e^{irT} \right) \right) \\
&= \frac{-i}{4} \sum_{r=1}^{\infty} r \left(\frac{1}{2} \sum_{n=1}^{r-1} \left(P_{r-n} P_n e^{-irT} - P_{r-n}^* P_n^* e^{irT} \right) + \sum_{n=r+1}^{\infty} \left(P_{n-r}^* P_n e^{-irT} - P_{n-r} P_n^* e^{irT} \right) \right)
\end{aligned}$$

Inserting the appropriate derivatives and the nonlinear term into the modified form of the KZK equation gives

$$\begin{aligned}
\frac{1}{2} \left[\sum_{n=1}^{\infty} \frac{\partial P_n}{\partial \sigma} e^{-inT} + \frac{\partial P_n^*}{\partial \sigma} e^{inT} \right] &= \\
\frac{P_{ab}}{2} \left[\sum_{n=1}^{\infty} -n^2 \left(P_n e^{-inT} + P_n^* e^{inT} \right) \right] &+ \frac{i}{8(1+\sigma)^2} \left[\sum_{n=1}^{\infty} \frac{1}{n} \left(\nabla_X^2 P_n e^{-inT} - \nabla_X^2 P_n^* e^{inT} \right) \right] \\
- \frac{ip_{nonlin}}{4(1+\sigma)} \sum_{q=1}^{\infty} q \left[\frac{1}{2} \sum_{n=1}^{q-1} \left(P_{q-n} P_n e^{-iqT} - P_{q-n}^* P_n^* e^{iqT} \right) + \sum_{n=q+1}^{\infty} \left(P_{n-q}^* P_n e^{-iqT} - P_{n-q} P_n^* e^{iqT} \right) \right]
\end{aligned}$$

Writing $P_n = g_n + ih_n$ and equating $\sin(nT)$ and $\cos(nT)$ terms throughout leads to two sets of nonlinear partial differential equations:

$$\begin{aligned}
\frac{\partial g_n}{\partial \sigma} &= \frac{np_{nonlin}}{2(1+\sigma)} \left\{ \sum_{j=1}^{n-1} h_j g_{n-j} + \sum_{j=n+1}^{\infty} (g_{j-n} h_j - h_{j-n} g_j) \right\} - n^2 p_{abs} g_n - \frac{\nabla_X^2 h_n}{4n(1+\sigma)^2}, \\
\frac{\partial h_n}{\partial \sigma} &= \frac{np_{nonli}}{2(1+\sigma)} \left\{ \frac{1}{2} \sum_{j=1}^{n-1} (h_j h_{n-j} - g_j g_{n-j}) - \sum_{j=n+1}^{\infty} (g_j g_{j-n} + h_j h_{j-n}) \right\} - n^2 p_{abs} h_n + \frac{\nabla_X^2 g_n}{4n(1+\sigma)^2}, \\
n &= 1, 2, \dots, \infty
\end{aligned}$$

Truncating this series after N terms and imposing limits on the summations over j leads to the equations solved by the software.

ANNEX B – SOFTWARE MANUAL

This document describes the workings of the Fortran executable kzkcy107.exe, which solves the KZK equation for an axisymmetric geometry.

The software solves the equations

$$\begin{aligned}\frac{\partial g_n}{\partial \sigma} &= \frac{np_{nonlin}}{2(1+\sigma)} \left\{ \sum_{j=1}^{n-1} h_j g_{n-j} + \sum_{j=n+1}^N (g_{j-n} h_j - h_{j-n} g_j) \right\} - n^2 p_{abs} g_n - \frac{\nabla_X^2 h_n}{4n(1+\sigma)^2}, \\ \frac{\partial h_n}{\partial \sigma} &= \frac{np_{nonli}}{2(1+\sigma)} \left\{ \frac{1}{2} \sum_{j=1}^{n-1} (h_j h_{n-j} - g_j g_{n-j}) - \sum_{j=n+1}^N (g_j g_{j-n} + h_j h_{j-n}) \right\} - n^2 p_{abs} h_n + \frac{\nabla_X^2 g_n}{4n(1+\sigma)^2}, \\ n &= 1, 2, \dots, N\end{aligned}$$

where ∇_X^2 denotes the radial term in the Laplacian operator.

These equations are derived from setting

$$\begin{aligned}P &= \sum_{n=1}^N [g_n(X, \sigma) \cos(nT) + h_n(X, \sigma) \sin(nT)] \\ &= \frac{1}{2} \left[\sum_{n=1}^N P_n(X, \sigma) e^{-inT} + \sum_{n=1}^N P_n^*(X, \sigma) e^{inT} \right] \\ &= \text{Real} \left[\sum_{n=1}^N P_n(X, \sigma) e^{-inT} \right]\end{aligned}$$

where P^* is the complex conjugate of P , in the equation

$$\frac{\partial P}{\partial \sigma} = \frac{p_{nonlin}}{(1+\sigma)} P \frac{\partial P}{\partial T} + p_{abs} \frac{\partial^2 P}{\partial T^2} + \frac{1}{4(1+\sigma)^2} \int_{-\infty}^T \nabla_X^2 P(X, \sigma, \hat{T}) d\hat{T}.$$

This last equation can be derived from the KZK equation

$$\frac{\partial p}{\partial z} = \frac{\beta p}{\rho_0 c_0^3} \frac{\partial p}{\partial \tau} + \frac{\delta}{2c_0^3} \frac{\partial^2 p}{\partial \tau^2} + \frac{c_0}{2} \int_{-\infty}^{\tau} \nabla_{\perp}^2 p(r, z, \hat{\tau}) d\hat{\tau}$$

by using the substitutions

$$\sigma = \frac{z}{z_0}, \quad X = \frac{r/a}{1+z/z_0}, \quad T = \omega_0 \tau - \frac{r^2/a^2}{1+z/z_0}, \quad P = \frac{p(r, z, \tau)(1+z/z_0)}{p_0},$$

where $z_0 = \omega_0 a^2 / 2c_0$. Using these substitutions leads to

$$p_{ab} = \frac{\beta p_0 \omega_0^2 a^2}{2\rho_0 c_0^4}, \quad p_{nonlin} = \frac{\delta a^2 \omega_0^3}{4c_0^4}.$$

The software requires the user to supply the following values:

$$c_0, p_0, a, f_0, \rho_0, \delta, \beta,$$

where $\omega_0 = 2 \pi f_0$, along with various quantities that define the output, the main axial step size, and the initial conditions. It is assumed that the initial conditions are measured on $z = 0$ in the form

$$p = q(r)\cos\omega_0 t,$$

where $q(r)$ is a complex-valued function supplied by the user. The boundary conditions used in the file are calculated from the real and imaginary parts of q . If $q(r) = p_0(A(r) + iB(r))$ then the conditions as implemented are

$$\begin{aligned} g_1(R,0) &= A(aR)\cos(R^2) + B(aR)\sin(R^2) \\ h_1(R,0) &= -A(aR)\sin(R^2) + B(aR)\cos(R^2) \\ g_n(R,0) &= h_n(R,0) = 0, \quad n = 2,3,\dots,M \end{aligned}$$

The output quantities are: on-axis pressure at user-specified times, radial pressure distributions for user specified values of time and z , full list of on-axis harmonics, radial distributions for each harmonic at user specified values of z .

Two axial steps are used in the calculation. In the initial stages, when the diffusion term dominates, a quasi-implicit method is used (quasi because the nonlinear term is calculated using the previous step's values so that no iteration is required) using a time step that avoids the Gibbs phenomena that can be associated with the problem. The later stages use a user-defined axial step to ensure that the stability problems caused by the nonlinearity beginning to dominate can be avoided. As yet, no analysis has been carried out to provide guidance regarding suitable time steps, so a trial and error approach must be taken.

The input file

The input file expects the following values:

- c_0 , the speed of sound,
- p_0 , the characteristic pressure amplitude of the source,
- a , the characteristic radius of the source,
- f_0 , the frequency of the source,
- ρ_0 , the density of the medium,
- δ , the sound diffusivity,
- β , the coefficient of nonlinearity,
- $stopz$, the maximum value of z that the calculation will run to,
- n_{out} , the number of output points of the radial distributions along the z axis
- $\{z_j, j = 1, 2, \dots, n_{out}\}$, the locations of the points on the axis at which the radial pressure distribution is required,
- N_{zout} , the number of points at which to output the on-axis results,
- z_{min} , the minimum value of z at which the on-axis values are required,

- z_{max} , the maximum value of z at which on-axis values are required,
- n_t , the number of times at which the pressure distribution is required
- M , the number of harmonics used in the summation, (maximum of 1000)
- n_r , the number of radial measurements used in the boundary conditions,
- $\{r_i, \text{Re}\{F(r_i)\}, \text{Im}\{F(r_i)\}, i = 1, 2, \dots, n_r\}$, the measurements for the boundary conditions,
- r_{max} , the size of the domain in the r direction at $z = 0$,
- N_r , the number of points to use in the radial direction (max 1000, more points improves results quality but increases run time, recommended min 255),
- dz , the main step size in the axial direction, in the same units as all other lengths.
-

As z increases, the size of the domain in the r direction increases because the dimensionless radial coordinate is $\xi = (r/a)/(1 + z/z_0)$, and so at an axial distance z_1 , the maximum radius is $r_{max}(1 + z_1/z_0)$.

Note that if either z_{max} or the final value of the z_j is larger than *stopz*, a warning will be issued and you will be offered the choice to continue the calculation by pressing a number or stop it by pressing a letter. This enables any mistakes in the input file to be corrected. Also note that the on-axis pressure will be output at the N_{zout} values $\{z_{min} + j^*(z_{max} - z_{min})/(N_{zout} - 1); j = 0, 1, 2, \dots, N_{zout} - 1\}$.

Part of a typical input file is shown below.

```

1503.00          C0
6.5699e+005      p0
0.0381          a
470000          f0
1000.0          ρ0
4.3605e-006     δ
3.5             β
2              nout
0              z1
6.4173e-001     z2
2              nt
12             M
52             nr
0              6.5699e+005  0.0000e+000  r1, Re{F(r1)},
Im{F(r52)}
7.4706e-004     6.5699e+005  0.0000e+000  r2, Re{F(r2)},
Im{F(r52)}
1.4941e-003     6.5699e+005  0.0000e+000  r3, Re{F(r3)},
Im{F(r52)}
- snip -
3.7353e-002     6.5699e+005  0.0000e+000  r51, Re{F(r51)},
Im{F(r52)}
3.8100e-002     6.5699e+005  0.0000e+000  r52, Re{F(r52)},
Im{F(r52)}
3.0480e-001     rmax
400             Nr
200             Nzout
1.0e-4          dz

```

The total number of output files will be $2*n_t + n_{out} + 1$, consisting of n_t files with the pressure versus z at n_{zout} points, n_t files with the pressure versus r , n_{out} files containing a full list of the amplitudes and phases of the harmonics at each radial position at the points z_j , $j = 1, 2, \dots, n_{out}$, and one file containing the on-axis values of all the harmonics at n_{zout} points.

The output files are named either Axi nnn .txt, Rad nnn .txt, or Harm $mmmm$.txt, where nnn is a three digit number (possibly starting with a zero) and $mmmm$ is a four-digit number, or R0harms.txt. Each of the Axi- and Rad- files represents the pressure in a region at a different time as specified by the user. The user specifies the number of output times, n_t , and the results are then output at times $\{(nnn - 1)/(f_0.n_t), nnn = 1, 2, \dots, n_t\}$. The Axi- files contain the pressure distribution along the axis ($r = 0$), and the Rad- files contain the radial distribution of pressure at the user-defined values of z . In Rad- and Axi- files the pressure is given as the real value as given by the sum in equation(1). The beginning of a typical Axi- file is shown below. In this case, $f_0 = 13.9$ kHz and $n_t = 2$, so $t_2 = 3.597 \times 10^{-5}$ s.

```
TIME = 1.063829787234043E-006
      Z          p
0.000000E+00 -6.569900E+05
3.208650E-03 -6.567288E+05
6.417300E-03 -6.559683E+05
9.625950E-03 -6.547107E+05
```

The Harm- files contain the harmonics G_n and H_n used in the sum in equation (1). These are not the same as the functions g_n and h_n used in the calculations. In fact,

$$G_n(\xi, \sigma) + iH_n(\xi, \sigma) = p_0 (g_n(\xi, \sigma) + ih_n(\xi, \sigma)) e^{iK} / (1 + \sigma),$$

$$K = \xi^2 (1 + \sigma).$$

A separate file is created for each value of z where the user has requested results, and each file contains all the harmonics at all N_r values of the radial coordinate. If there are a large number of harmonics, these files can be very large. The beginning of a typical file is shown below. The column headed "Real part" contains values of G_n , and the column headed "Imag part" contains values of H_n .

```
Harmonics for r = 0.0000E+00 z = 1.1757E-01
      N      Real part      Imag part
1  -8.394676E+05    2.061647E+06
2  -8.400115E+05    5.177273E+05
3  -5.852783E+05    2.508181E+05
4  -2.006985E+06   -1.003354E+04
```

The file R0harms.txt contains the real and imaginary parts of all the harmonics along $r = 0$. Each line has three values: z , $\text{Re}(p_n)$, $\text{Im}(p_n)$. No further formatting is given for ease of reading into other applications. The start of a typical file is shown below.

```

0.000000E+00    6.569900E+05    0.000000E+00
0.000000E+00    0.000000E+00    0.000000E+00
0.000000E+00    0.000000E+00    0.000000E+00
0.000000E+00    0.000000E+00    0.000000E+00
0.000000E+00    0.000000E+00    0.000000E+00
0.000000E+00    0.000000E+00    0.000000E+00
0.000000E+00    0.000000E+00    0.000000E+00
0.000000E+00    0.000000E+00    0.000000E+00
0.000000E+00    0.000000E+00    0.000000E+00
3.208650E-03    6.568277E+05    1.391061E+04
3.208650E-03    8.923141E+01   -2.105706E+03
3.208650E-03   -9.677634E+00   -6.156109E-01
3.208650E-03   -4.346332E-03    5.119070E-02

```