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The solution of ordinary differential equations arising from stress transfer mechanics of general symmetric laminates

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ABSTRACT

Stress transfer for general symmetric multiple-ply laminates has been modelled by a technique whereby each ply, which may be oriented in any direction preserving symmetry about the mid-plane of the laminate, is represented by a series of parallel ply elements. The planes are assumed to be perfectly bonded together and the case has been considered where loading causes transverse cracks to appear in some or all of the plies having a single specified orientation. This model generates a pair of coupled systems of ordinary differential equations with appropriate boundary conditions. One system of differential equations is fourth order and the other is second order. The purpose of this report is to describe a semi-analytic method for solving these differential equations.

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CONTENTS

1 INTRODUCTION

This report describes a semi-analytical method for the solution of a pair of coupled systems of simultaneous ordinary differential equations arising from the analysis of the stress transfer mechanics of a general symmetric multiple-ply laminate. The stress transfer in laminates is characterised by functions $p_i(y), q_i(y)$, $i = 1, 2, \ldots, N$, where N is the number of interfaces between ply elements. One system is fourth order in the variables $p_i(y)$ and second order in the variables $q_i(y)$ and the other system is second order in both the $p_i(y)$ and the $q_i(y)$.

In Section 2, we introduce the differential equations and boundary conditions which constitute the problem. The auxiliary/characteristic equation, which determines eigensolutions of the equations, is explained in Section 3. Once this equation has been solved, the solution functions, $p_i(y)$, $q_i(y)$ are known to within a set of arbitrary constants which are to be determined by the boundary conditions.

In Section 4, we show that, provided the eigensolutions of the characteristic equation are all distinct, each of the $p_i(y)$, $q_i(y)$ functions can be expressed analytically in the form of series expansions. Using the symmetry of the problem about the plane $y = 0$ (boundary conditions and formulation), we show in Section 5 that many of the coefficients to be determined are not independent of each other and so the series expansions may be considerably reduced in size.

Since the ordinary differential equations are linear with constant coefficients, we demonstrate in Section 6 that the arbitrary coefficients for each of the solution functions are interrelated. We discuss a method which is used to determine the relationships between these sets of coefficients. The result of using this procedure is that arbitrary coefficients only need to be found for *one* of the solution functions. We refer to these as the *baseline* arbitrary constants.

In Section 7, we present the form of various derivatives and integrals of the solution functions in terms of the series expansions and indicate how we overcome a potential problem involving the numerical evaluation of exponential functions with large arguments. The expressions for these derivatives and integrals are used to set up a system matrix and right hand side vector which are equivalent to a statement of the boundary conditions in matrix form.

We then explain how subsystem matrices are developed which correspond to boundary conditions of each of a number of different types. These subsytems are incorporated, in Section 8, into a global system of simultaneous linear algebraic equations, which are then solved for the baseline arbitrary constants.

During the complete solution process, some standard linear algebra techniques are employed. For these, we use (and make reference to in this report) mathematical software modules of the highest quality. For example, use is made of the EISPACK Fortran library module QZVAL for computing solutions of the generalised eigenvalue problem outlined in Section 3.

We indicate in Section 9 that the solution functions are all real-valued, despite the presence of complex-valued arbitrary constants and eigensolutions.

Finally, in Section 10, we provide a worked example with corresponding results.

2 THE DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

Following McCartney [1], the fourth order system of N differential equations for $N+1$ ply elements (N ply interfaces, $x = x_i, i = 1, \ldots, N$) is written

$$\sum_{i=1}^{N} F_{i,j} p_i^{(4)}(y) + \sum_{i=1}^{N} G_{i,j} p_i^{(2)}(y) + \sum_{i=1}^{N} H_{i,j} p_i(y)$$
$$+ \sum_{i=1}^{N} g_{i,j} q_i^{(2)}(y) + \sum_{i=1}^{N} h_{i,j} q_i(y) = 0, \quad j = 1, 2, \ldots, N \tag{1}$$

and the second order system is

$$\sum_{i=1}^{N} A_{i,j} p_i^{(2)}(y) + \sum_{i=1}^{N} B_{i,j} p_i(y)$$
$$+ \sum_{i=1}^{N} a_{i,j} q_i^{(2)}(y) + \sum_{i=1}^{N} b_{i,j} q_i(y) = 0, \quad j = 1, 2, \ldots, N, \tag{2}$$

for $y \in [-L, L]$, where $2L$ is the length of laminate considered and where the matrices $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{a}$ and $\mathbf{b}$ are constant coefficients which are derived from recurrence relations. The $p_i(y)$ and $q_i(y)$ are the unknowns to be determined.

The first system requires $4N$ boundary conditions and the second needs $2N$, making $6N$ in total. Additionally, for this problem, attention is restricted to the case where there is symmetry about the plane $y = 0$. Hence, only $3N$ boundary conditions are required.

The first N boundary conditions are that the first derivatives of the $p_i(y)$ functions vanish for each ply element interface thus:

$$p_i'(L) = 0, \quad i = 1, \ldots, N. \tag{3}$$

Suppose that there are c cracked and u uncracked ply elements. Then, clearly $c + u = N + 1$ as N is the number of interfaces between elements. Suppose further that $I_p = \{1, 2, \ldots, N+1\}$ is the set of all ply elements, $I_c \subset I_p$ is the set of indices of the cracked ply elements and $I_u \subset I_p$ the set of indices of the uncracked ply elements. Then, we have $I_p = I_c \bigcup I_u$ and $I_c \bigcap I_u \equiv 0$.

For each of the *cracked* ply elements, the following two conditions hold:

$$p_i(L) - p_{i-1}(L) = h_i \sigma_i, \quad i \in I_c, \tag{4}$$

$$q_i(L) - q_{i-1}(L) = h_i \tau_i, \quad i \in I_c. \tag{5}$$

There are clearly $2c$ of these boundary conditions.

For the remaining boundary conditions, we identify the *nearest uncracked ply element to the centre plane of the laminate* and assign its ply element number to the integer variable j. Then, for each of the *uncracked* ply elements, except ply element number j, the following two conditions hold:

$$\frac{\overline{g}_{22}^{i}}{h_i L}\left[p_{i-1}^*(L) - p_i^*(L)\right] - \frac{\overline{g}_{22}^{j}}{h_j L}\left[p_{j-1}^*(L) - p_j^*(L)\right]$$
$$+ \frac{\overline{g}_{24}^{i}}{h_i L}\left[q_{i-1}^*(L) - q_i^*(L)\right] - \frac{\overline{g}_{24}^{j}}{h_j L}\left[q_{j-1}^*(L) - q_j^*(L)\right] = 0, \quad i \in \{I_u \setminus j\}, \tag{6}$$

$$\frac{\overline{g}^{i}_{24}}{h_i L}\left[p^*_{i-1}(L) - p^*_i(L)\right] - \frac{\overline{g}^{j}_{24}}{h_j L}\left[p^*_{j-1}(L) - p^*_j(L)\right]$$
$$+\frac{\overline{g}^{i}_{44}}{h_i L}\left[q^*_{i-1}(L) - q^*_i(L)\right] - \frac{\overline{g}^{j}_{44}}{h_j L}\left[q^*_{j-1}(L) - q^*_j(L)\right] = 0 \quad i \in \{I_u \setminus j\}. \tag{7}$$

There are $2u - 2$ of these boundary conditions, giving

$$N + 2c + (2u - 2) = N + 2(c + u - 1) = 3N$$

in total, as required. For boundary conditions (4) to (7), h_i is the thickness of the ith ply element and the values σ_i, τ_i and the array $\overline{g}$ are all provided by the modelling software of McCartney [1].

It should be noted that when the general symmetric ply model is applied to a cross-ply laminate (i.e. the plies are either parallel or perpendicular to one another), the coefficients g_{ij}, h_{ij}, A_{ij} and B_{ij} all vanish with the result that the simultaneous set of differential equations given by (1) and (2) separates into two independent systems of equations. The numerical solution technique described in this report cannot be used to solve the differential equations which result in the case of cross-ply laminates. The modelling for cross-ply laminates under biaxial loading is described in (McCartney and Saunderson, 1995)[2] and that for cross-ply laminates subject to shear loading in (McCartney and Ferriss, 1995) [3]. The differential equation solver which is to be used in the cross-ply case is described in (Hannaby, 1993)[4].

3 THE AUXILIARY/CHARACTERISTIC EQUATION

We may express system (1), using matrix and differential operator notation, as

$$\mathbf{F}^{\mathrm{T}}\mathcal{D}^4\mathbf{p} + \mathbf{G}^{\mathrm{T}}\mathcal{D}^2\mathbf{p} + \mathbf{H}^{\mathrm{T}}\mathbf{p} + \mathbf{g}^{\mathrm{T}}\mathcal{D}^2\mathbf{q} + \mathbf{h}^{\mathrm{T}}\mathbf{q} = \mathbf{0} \tag{8}$$

and for system (2) similarly:

$$\mathbf{A}^{\mathrm{T}}\mathcal{D}^2\mathbf{p} + \mathbf{B}^{\mathrm{T}}\mathbf{p} + \mathbf{a}^{\mathrm{T}}\mathcal{D}^2\mathbf{q} + \mathbf{b}^{\mathrm{T}}\mathbf{q} = \mathbf{0}, \tag{9}$$

where $\mathbf{p} = (p_1(y), \ldots, p_N(y))^{\mathrm{T}}$ and $\mathbf{q} = (q_1(y), \ldots, q_N(y))^{\mathrm{T}}$. Combining these systems, we obtain the coupled system, expressed as follows:

$$\begin{pmatrix} \mathbf{F}^{\mathrm{T}}\mathcal{D}^4 + \mathbf{G}^{\mathrm{T}}\mathcal{D}^2 + \mathbf{H}^{\mathrm{T}} & \mathbf{g}^{\mathrm{T}}\mathcal{D}^2 + \mathbf{h}^{\mathrm{T}} \\ \mathbf{A}^{\mathrm{T}}\mathcal{D}^2 + \mathbf{B}^{\mathrm{T}} & \mathbf{a}^{\mathrm{T}}\mathcal{D}^2 + \mathbf{b}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \mathbf{0}. \tag{10}$$

Let us denote the differential operator matrix on the left hand side of equation (10) by $\mathbf{M}(\mathcal{D})$, i.e.,

$$\mathbf{M}(\mathcal{D}) = \begin{pmatrix} \mathbf{F}^{\mathrm{T}}\mathcal{D}^4 + \mathbf{G}^{\mathrm{T}}\mathcal{D}^2 + \mathbf{H}^{\mathrm{T}} & \mathbf{g}^{\mathrm{T}}\mathcal{D}^2 + \mathbf{h}^{\mathrm{T}} \\ \mathbf{A}^{\mathrm{T}}\mathcal{D}^2 + \mathbf{B}^{\mathrm{T}} & \mathbf{b}^{\mathrm{T}}\mathcal{D}^2 + \mathbf{b}^{\mathrm{T}} \end{pmatrix}$$

Then, the characteristic equation for the coupled system is

$$|\mathbf{M}(\lambda)| = \det\left\{\begin{pmatrix} \mathbf{F}^{\mathrm{T}}\lambda^4 + \mathbf{G}^{\mathrm{T}}\lambda^2 + \mathbf{H}^{\mathrm{T}} & \mathbf{g}^{\mathrm{T}}\lambda^2 + \mathbf{h}^{\mathrm{T}} \\ \mathbf{A}^{\mathrm{T}}\lambda^2 + \mathbf{B}^{\mathrm{T}} & \mathbf{a}^{\mathrm{T}}\lambda^2 + \mathbf{b}^{\mathrm{T}} \end{pmatrix}\right\} = 0. \tag{11}$$

The characteristic equation (11) is a polynomial of degree $6N$ in λ. Since the derivatives in the coupled system are all even, the λ_i, $i = 1, 2, \ldots, 6N$ appear in positive- and negative-valued pairs of square-roots, i.e., they may be arranged so that $\lambda_i = -\lambda_{i+3N}$, $i = 1, 2, \ldots, 3N$. Having arranged the λ_i, which could be complex-valued, in this way, we may set $\xi_i = \lambda_i^2$ for each λ_i, $i = 1, \ldots, 3N$.

As we shall see in Section 4, the functions $p_i(y)$ and $q_i(y)$ may be written as series expansions of exponential functions as long as the values λ_i are all distinct. In other words, each of these solution functions is of the form

$$f(y) = \sum_{i=1}^{6N} C_i \, e^{\lambda_i y}$$

and so we have, for the kth derivative, the relation

$$\mathcal{D}^k f(y) = \sum_{i=1}^{6N} C_i \, \lambda_i^k \, e^{\lambda_i y}.$$

Because of this property of linear systems of ordinary differential equations with constant coefficients, solving the characteristic equation (11) is equivalent to finding roots, λ_i, $i = 1, \ldots, 6N$ and solution functions $\mathbf{p}, \mathbf{q}$ which satisfy

$$\mathrm{M}(\lambda) \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \mathbf{0}.$$

Using the ξ-values defined above, we now present this equation in full; find ξ_i, $i = 1, \ldots, 3N$, $\mathbf{p} = (p_1(y), \ldots, p_N(y))^{\mathrm{T}}$ and $\mathbf{q} = (q_1(y), \ldots, q_N(y))^{\mathrm{T}}$ which are solutions of

$$\begin{pmatrix} \mathbf{F}^{\mathrm{T}}\xi^2 + \mathbf{G}^{\mathrm{T}}\xi + \mathbf{H}^{\mathrm{T}} & \mathbf{g}^{\mathrm{T}}\xi + \mathbf{h}^{\mathrm{T}} \\ \mathbf{A}^{\mathrm{T}}\xi + \mathbf{B}^{\mathrm{T}} & \mathbf{a}^{\mathrm{T}}\xi + \mathbf{b}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \mathbf{0}.$$

Now let us make the definition

$$\mathbf{r} = \xi \mathbf{p}.$$

Then, we have, in terms of ξ,

$$\mathbf{H}^{\mathrm{T}}\mathbf{p} + \mathbf{h}^{\mathrm{T}}\mathbf{q} = -\xi\mathbf{F}^{\mathrm{T}}\mathbf{r} - \xi\mathbf{G}^{\mathrm{T}}\mathbf{p} - \xi\mathbf{g}^{\mathrm{T}}\mathbf{q},$$

$$\mathbf{B}^{\mathrm{T}}\mathbf{p} + \mathbf{b}^{\mathrm{T}}\mathbf{q} = -\xi\mathbf{A}^{\mathrm{T}}\mathbf{p} - \xi\mathbf{a}^{\mathrm{T}}\mathbf{q}.$$

Expressing these two equations with the above definition of $\mathbf{r}$, we get, in matrix form:

$$\begin{pmatrix} \mathbf{H}^{\mathrm{T}} & \mathbf{h}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{b}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{r} \end{pmatrix} = \xi \begin{pmatrix} -\mathbf{G}^{\mathrm{T}} & -\mathbf{g}^{\mathrm{T}} & -\mathbf{F}^{\mathrm{T}} \\ -\mathbf{A}^{\mathrm{T}} & -\mathbf{a}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{r} \end{pmatrix} \tag{12}$$

This is the *generalised eigenvalue problem* form of the characteristic equation, which is the problem actually solved in this work. The numerical solution of a $6N$th degree polynomial of type (11) is unstable, particularly for large values of N. The generalised eigenvalue problem (12) is better conditioned. Here, equation (12) is solved using the EISPACK Fortran subroutine QZVAL [5, 6]. This software is available as a public domain module and is chosen because of the good pedigree of the EISPACK library. We note further that subroutine QZVAL forms the basis of Matlab's [7] EIG function.

4 SERIES EXPANSIONS FOR THE SOLUTION FUNCTIONS

The generalised eigenvalue problem (12) is solved to yield ξ_i, $i = 1, 2, \ldots, 3N$ and hence λ_i, $i = 1, 2, \ldots, 6N$. Then we may write, without loss of generality,

$$p_k(y) = \sum_{i=1}^{6N} A_{i,k} \, e^{\lambda_i y} \tag{13}$$

$$q_k(y) = \sum_{i=1}^{6N} A_{i,N+k}\, e^{\lambda_i y}, \tag{14}$$

for $k = 1, 2, \ldots, N$. This is because the system is one of linear ordinary differential equations with constant coefficients. It is necessary that all the λ_i values are *distinct* for expressions (13), (14) to hold. This uniqueness condition is checked after solving (12) and is found to be satisfied by the input data in all cases considered. If there were any repeated eigenvalues, then the form of the finite sums in (13) and (14) and the methodology described in this report would not be applicable.

It will become clear later in Section 6 that the values $A_{i,2N}$ in equation (14) are the ***arbitrary constants*** for function $q_N(y)$ and the $A_{i,2N-1}$ are the arbitrary constants for $q_{N-1}(y)$. There are $2N$ functions under consideration and we use the notation $A_{i,2N}$ for the arbitrary constants for $q_N(y)$ because this is the $2N$th solution function.

5 REDUCTION OF THE SERIES EXPANSIONS

Since only $3N$ of the eigenvalues λ_i are independent (the complete $6N$ of these appear in positive and negative pairs), and from the symmetry of the problem (boundary conditions and formulation), it follows that only $3N$ of the arbitrary constants for each of the solution functions are independent. In (13), the coefficient of $e^{\lambda_i y}$ must match that of $e^{\lambda_{i+3N} y}$ for each of $i = 1, 2, \ldots, 3N$. For example, for function $q_N(y)$, we find that $A_{i,2N} = A_{i+3N,2N}, i = 1, 2, \ldots, 3N$ and for $q_{N-1}(y)$, $A_{i,2N-1} = A_{i+3N,2N-1}, i = 1, 2, \ldots, 3N$.

Therefore, the reduced series expansions for the $p_k(y)$ and $q_k(y)$ are

$$p_k(y) = 2\sum_{i=1}^{3N} A_{i,k} \cosh(\lambda_i y)$$

and

$$q_k(y) = 2\sum_{i=1}^{3N} A_{i,N+k} \cosh(\lambda_i y).$$

6 THE D-OPERATOR METHOD

Returning to the differential operator formulation of the coupled system (10), we see that the solution functions $p_i(y)$ and $q_i(y)$ are related to each other in the following way:

$$\mathrm{M}(\mathcal{D}) \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \mathbf{0}, \tag{15}$$

where

$$\begin{aligned}
M_{i,j}(\mathcal{D}) &= F_{j,i}\mathcal{D}^4 + G_{j,i}\mathcal{D}^2 + H_{j,i}, \\
M_{i+N,j}(\mathcal{D}) &= A_{j,i}\mathcal{D}^2 + B_{j,i}, \\
M_{i,j+N}(\mathcal{D}) &= g_{j,i}\mathcal{D}^2 + h_{j,i}, \\
M_{i+N,j+N}(\mathcal{D}) &= a_{j,i}\mathcal{D}^2 + b_{j,i},
\end{aligned}$$

for $i, j = 1, 2, \ldots, N$.

For convenience, we make the following definition of the matrix operator $\mathbf{M}$:

$$\mathbf{M}(x) = \begin{pmatrix} \mathbf{F}^{\mathrm{T}}x^4 + \mathbf{G}^{\mathrm{T}}x^2 + \mathbf{H}^{\mathrm{T}} & \mathbf{g}^{\mathrm{T}}x^2 + \mathbf{h}^{\mathrm{T}} \\ \mathbf{A}^{\mathrm{T}}x^2 + \mathbf{B}^{\mathrm{T}} & \mathbf{a}^{\mathrm{T}}x^2 + \mathbf{b}^{\mathrm{T}} \end{pmatrix}.$$

Now consider the application of $\mathbf{M}(\mathcal{D})$ to the kth term of each of the solution functions. Then we have

$$\mathbf{M}(\mathcal{D}) \begin{pmatrix} 2\theta_k\, A_{k,1} \\ 2\theta_k\, A_{k,2} \\ \vdots \\ 2\theta_k\, A_{k,2N} \end{pmatrix} = \begin{pmatrix} \mathbf{F}^{\mathrm{T}}\mathcal{D}^4 + \mathbf{G}^{\mathrm{T}}\mathcal{D}^2 + \mathbf{H}^{\mathrm{T}} & \mathbf{g}^{\mathrm{T}}\mathcal{D}^2 + \mathbf{h}^{\mathrm{T}} \\ \mathbf{A}^{\mathrm{T}}\mathcal{D}^2 + \mathbf{B}^{\mathrm{T}} & \mathbf{a}^{\mathrm{T}}\mathcal{D}^2 + \mathbf{b}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 2\theta_k\, A_{k,1} \\ 2\theta_k\, A_{k,2} \\ \vdots \\ 2\theta_k\, A_{k,2N} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{F}^{\mathrm{T}}\lambda_k^4 + \mathbf{G}^{\mathrm{T}}\lambda_k^2 + \mathbf{H}^{\mathrm{T}} & \mathbf{g}^{\mathrm{T}}\lambda_k^2 + \mathbf{h}^{\mathrm{T}} \\ \mathbf{A}^{\mathrm{T}}\lambda_k^2 + \mathbf{B}^{\mathrm{T}} & \mathbf{a}^{\mathrm{T}}\lambda_k^2 + \mathbf{b}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} 2\theta_k\, A_{k,1} \\ 2\theta_k\, A_{k,2} \\ \vdots \\ 2\theta_k\, A_{k,2N} \end{pmatrix} = \mathbf{M}(\lambda_k) \begin{pmatrix} 2\theta_k\, A_{k,1} \\ 2\theta_k\, A_{k,2} \\ \vdots \\ 2\theta_k\, A_{k,2N} \end{pmatrix}, \quad (16)$$

where $\theta_k = \cosh(\lambda_k y)$.

Let us now make the following definitions

$$\mathbf{A}_k = \begin{pmatrix} A_{k,1} \\ A_{k,2} \\ \cdot \\ A_{k,2N} \end{pmatrix}$$

and

$$\mathbf{E}_k = 2\theta_k\, \mathbf{I},$$

where $\mathbf{I}$ is the identity matrix of order $2N$. Then, we may write equation (16) as follows:

$$\mathbf{M}(\mathcal{D})\, \mathbf{E}_k\, \mathbf{A}_k = \mathbf{M}(\lambda_k)\, \mathbf{E}_k\, \mathbf{A}_k. \tag{17}$$

We may express the complete set of solution functions, using this notation, as

$$\begin{pmatrix} p_1(y) \\ \\ p_N(y) \\ q_1(y) \\ \vdots \\ q_N(y) \end{pmatrix} = \mathbf{E}_1\, \mathbf{A}_1 + \mathbf{E}_2\, \mathbf{A}_2 + \ \ldots + \mathbf{E}_{3N}\, \mathbf{A}_{3N}.$$

Since equation (17) holds for *any* value of k, we may rewrite equation (15) as

$$\mathbf{M}(\mathcal{D}) \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \mathbf{M}(\mathcal{D}) \left(\sum_{i=1}^{3N} \mathbf{E}_i\, \mathbf{A}_i \right) = \sum_{i=1}^{3N} \mathbf{M}(\lambda_i)\, \mathbf{E}_i\, \mathbf{A}_i = \mathbf{0}. \tag{18}$$

Each of the terms in the summation must ., thus ensuring that (18) is satisfied We evaluate the matrix

$$\mathbf{M}_k = \mathbf{M}(\lambda_k) = \begin{pmatrix} \mathbf{F}^{\mathrm{T}}\lambda_k^4 + \mathbf{G}^{\mathrm{T}}\lambda_k^2 + \mathbf{H}^{\mathrm{T}} & \mathbf{g}^{\mathrm{T}}\lambda_k^2 + \mathbf{h}^{\mathrm{T}} \\ \mathbf{A}^{\mathrm{T}}\lambda_k^2 + \mathbf{B}^{\mathrm{T}} & \mathbf{a}^{\mathrm{T}}\lambda_k^2 + \mathbf{b}^{\mathrm{T}} \end{pmatrix}, \tag{19}$$

for each $k = 1, 2, \quad , 3N$ and need to solve for $\mathbf{A}_k$ satisfying

$$\mathbf{M}_k \, \mathbf{E}_k \, \mathbf{A}_k = \mathbf{0}.$$

The LU decomposition of $\mathbf{M}_k$ is sought and the above equation may be rewritten as

$$\mathbf{L}_k \, \mathbf{U}_k \, \mathbf{E}_k \, \mathbf{A}_k = \mathbf{0}.$$

Since $\mathbf{L}_k$ is invertible and the system is homogeneous, we may premultiply the equation by the inverse of $\mathbf{L}_k$ to obtain

$$\mathbf{U}_k \, \mathbf{E}_k \, \mathbf{A}_k = \mathbf{0}.$$

Recall that $\mathbf{E}_k$ is simply a nonzero constant multiplied by an identity matrix, and so we may write

$$\mathbf{U}_k \, \mathbf{A}_k = \mathbf{0}.$$

For convenience, let us now refer to the matrix $\mathbf{U}_k$ as $\mathbf{U}^k$. This upper triangular form provides the following linear relationships between the arbitrary constants:

$$U^k_{2N-1,2N-1} \, A_{k,2N-1} + U^k_{2N-1,2N} A_{k,2N} = 0,$$

$$U^k_{2N-2,2N-2} \, A_{k,2N-2} + U^k_{2N-2,2N-1} \, A_{k,2N-1} + U^k_{2N-2,2N} \, A_{k,2N} = 0.$$

Therefore, from the first of these equations, we may find $A_{k,2N-1}$ in terms of $A_{k,2N}$ as

$$A_{k,2N-1} = -\frac{U^k_{2N-1,2N} A_{k,2N}}{U^k_{2N-1,2N-1}}.$$

All the other values of $A_{k,i}, i = 2N-2, 2N-3, \ldots, 1$ may be found similarly by the process of *backsubstitution.* This means that the arbitrary constants for $p_1(y), \ldots, p_N(y)$ and $q_1(y), \ldots, q_{N-1}(y)$ are related to those of $q_N(y)$ and may be obtained by the above process. We may therefore rename the arbitrary constants for $q_N(y)$, namely $A_{i,2N}, i = 1, 2, \ldots, 3N$ as $A_i, i = 1, 2, \ldots, 3N$. Thus, the function $q_N(y)$ may be expressed as

$$q_N(y) = 2 \sum_{i=1}^{3N} A_i \cosh(\lambda_i y)$$

and $q_{N-1}(y)$ as

$$q_{N-1}(y) = 2 \sum_{i=1}^{3N} A_i \mu_{i,1} \cosh(\lambda_i y),$$

where,

$$\mu_{i,1} = -\frac{U^i_{2N-1,2N}}{U^i_{2N-1,2N-1}}.$$

Function $q_{N-2}(y)$ may be expressed as

$$q_{N-2}(y) = \sum_{i=1}^{3N} A_i \mu_{i,2} \cosh(\lambda_i y),$$

where,

$$\mu_{i,2} = -\frac{U^i_{2N-2,2N-1} \mu_{i,1} + U^i_{2N-2,2N}}{U^i_{2N-2,2N-2}}.$$

If we make the definition $\mu_{i,0} = 1, i = 1, 2, \ldots, 3N$, then we may write generally

$$f_k(y) = 2 \sum_{i=1}^{3N} A_i \mu_{i,2N-k} \cosh(\lambda_i y), \tag{20}$$

where the first N of these functions are $p(y)$ functions and the last N are $q(y)$ functions. The above mentioned process of upper-triangularisation and backsubstitution is known as the *D-operator* method [8]. As for the generalised eigenvalue problem, we choose to use high-quality public-domain software to obtain the LU factorisation of the matrix M. The actual module employed in this work is the LINPACK Fortran [9] subroutine ZGEFA.

All the values of $\mu_{i,j}, j = 1,2,\ldots,2N-1$ may be found by applying the D-operator method to the matrix $\mathbf{M}_i$, defined in equation (19). By carrying out this process for *each* of the eigenvalues, and using the above definition of $\mu_{i,0}$, we obtain the $3N \times 2N$ matrix

$$\mu = \begin{pmatrix} 1 & \mu_{1,1} & & \mu_{1,2N-1} \\ 1 & \mu_{2,1} & & \mu_{2,2N-1} \\ \vdots & \vdots & & \vdots \\ 1 & \mu_{3N,1} & \cdots & \mu_{3N,2N-1} \end{pmatrix}$$

The entries in the kth column of this matrix are the factors by which we multiply the arbitrary constants A_i of the function $f_{2N}(y) \equiv q_N(y)$ in order to obtain the corresponding arbitrary constants $A_{i,2N-k+1}$ of the function $f_{2N-k+1}(y)$. For example, suppose $N = 5$ and so $f_8(y) \equiv q_3(y)$. Then the arbitrary constants for this function are the $A_{i,8}, i = 1,2,\ldots,3N$. Assuming the constants $A_i \equiv A_{i,10}$ are known for function $f_{10}(y) \equiv q_5(y)$, we may find the values of $A_{i,8}$ from the relationship

$$A_{i,8} = \mu_{i,2}\, A_i,$$

i.e., we use the elements of the third column of $\boldsymbol{\mu}$. Hence, there are only $3N$ arbitrary constants to be determined in the remaining part of the solution process, namely $A_i, i = 1,2,\ldots,3N$.

7 EVALUATION OF THE SOLUTION FUNCTIONS

We have established that the form of the solution functions is

$$f_k(y) = \sum_{i=1}^{3N} A_i\, \mu_{i,2N-k} \left(e^{\lambda_i y} + e^{-\lambda_i y}\right), \quad k = 1,2,\ldots,2N$$

In Section 3, we explained how the eigenvalues, λ_i are found and in Section 6, we saw how to determine the constant coefficient factors, $\mu_{i,j}$. The remaining problem is to compute the arbitrary constants, A_i. In order to achieve this, however, as we shall see in the next section, we shall need to evaluate all the exponential terms for each of the solution functions $f_k(y)$.

It should be clear that for very large values of $\lambda_i y$, the numerical evaluation of $e^{\lambda_i y}$ may cause computational overflow to arise. To overcome this problem, we choose to make the substitution

$$A_i^* = e^{\lambda_i L}\, A_i, \quad i = 1,2,\ldots,3N. \tag{21}$$

Then, we can write

$$f_k(y) = \sum_{i=1}^{3N} A_i^*\, \mu_{i,2N-k} \left(e^{\lambda_i (y-L)} + e^{-\lambda_i (y+L)}\right). \tag{22}$$

The boundary conditions involve the solution functions, their first derivatives and their definite integrals, evaluated at the point $y = L$. Also, in order to ultimately check the satisfaction of the differential equations, it will be necessary to be able to evaluate the solution functions and their second and fourth derivatives at a general point y. Therefore, we state the form of these functions here.

7.1 GENERAL FUNCTIONS

These are the functions f_k that need to be evaluated at a general point y.

The general derivatives

$$f_k^{(n)}(y) = \sum_{i=1}^{3N} A_i^* \mu_{i,2N-k} \lambda_i^n \left(e^{\lambda_i(y-L)} + (-1)^n e^{-\lambda_i(y+L)} \right)$$

This form applies for the functions themselves ($n = 0$) and the derivatives $n = 1, 2, 3, 4$.

The general definite integral The definite integral $f_k^*(y)$, which appears in the boundary conditions, is defined by

$$f_k^*(y) = \int_0^y f_k(s)\, ds.$$

Using the finite sum in (22), this may be written as follows:

$$f_k^*(y) = \sum_{i=1}^{3N} A_i^* \frac{\mu_{i,2N-k}}{\lambda_i} \left(e^{\lambda_i(y-L)} - e^{-\lambda_i(y+L)} \right)$$

7.2 FUNCTIONS EVALUATED AT $Y = L$

These are the finite sum representations appropriate for the boundary conditions, namely, the functions evaluated at $y = L$.

The derivatives

$$f_k^{(n)}(L) = \sum_{i=1}^{3N} A_i^* \mu_{i,2N-k} \lambda_i^n \left(1 + (-1)^n e^{-2\lambda_i L} \right)$$

The case $n = 0$ corresponds to the functions themselves.

The definite integral

$$f_k^*(L) = \sum_{i=1}^{3N} A_i^* \frac{\mu_{i,2N-k}}{\lambda_i} \left(1 - e^{-2\lambda_i L} \right)$$

8 INCLUSION OF THE BOUNDARY CONDITIONS

In this section, we consider how the boundary conditions, given by equations (3) to (7) are incorporated into a square system of linear algebraic equations, the unknowns of which are the arbitrary constants, A_i^*, $i = 1, 2, \ldots, 3N$ appearing in (22).

8.1 BOUNDARY CONDITIONS OF TYPE EQUATION (3)

These boundary conditions are of the form

$$f_i'(L) = 0, \quad i = 1, 2, \ldots, N.$$

(Recall that $f_i(y) \equiv p_i(y)$ and $f_{i+N}(y) \equiv q_i(y)$ for $i = 1, 2, \ldots, N$).

These boundary conditions may be written, using finite sum (25), as

$$\left[\begin{array}{ccc} \mu_{1,2N-1}\lambda_1\left(1-e^{-2\lambda_1 L}\right) & \ldots & \mu_{3N,2N-1}\lambda_{3N}\left(1-e^{-2\lambda_{3N} L}\right) \\ \vdots & & \\ \mu_{1,N}\lambda_1\left(1-e^{-2\lambda_1 L}\right) & & \mu_{3N,N}\lambda_{3N}\left(1-e^{-2\lambda_{3N} L}\right) \end{array}\right] \begin{pmatrix} A_1^* \\ \vdots \\ A_{3N}^* \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

It proves to be useful at this stage, to define the following matrices:

$$\Lambda = \begin{array}{cccc} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & . \\ 0 & & 0 & \lambda_{3N} \end{array}$$

$$\mathbf{E}^- = \begin{pmatrix} \left(1-e^{-2\lambda_1 L}\right) & 0 & & 0 \\ 0 & \left(1-e^{-2\lambda_2 L}\right) & & 0 \\ & & \vdots & \vdots \\ 0 & & 0 & \left(1-e^{-2\lambda_{3N} L}\right) \end{pmatrix}$$

$$\mathbf{E}^+ = \begin{pmatrix} \left(1+e^{-2\lambda_1 L}\right) & 0 & & 0 \\ 0 & \left(1+e^{-2\lambda_2 L}\right) & & 0 \\ . & & \vdots & . \\ 0 & & 0 & \left(1+e^{-2\lambda_{3N} L}\right) \end{pmatrix}$$

and

$$\mathbf{A}^* = \begin{pmatrix} A_1^* \\ \vdots \\ A_{3N}^* \end{pmatrix}$$

Then, if $\boldsymbol{\mu}_1$ is defined as the transpose of columns $[N+1 : 2N]$ of $\boldsymbol{\mu}$, these boundary conditions may be expressed as

$$\boldsymbol{\mu}_1 \, \Lambda \, \mathbf{E}^- \mathbf{A}^* = \mathbf{0}.$$

Consider the matrix

$$\Omega^1 = \boldsymbol{\mu}_1 \, \Lambda \, \mathbf{E}^-$$

and the null-vector $\mathbf{d}_1$ of length N Then the subsystem of boundary conditions of type (3) can be expressed as

$$\Omega^1 \, \mathbf{A}^* = \mathbf{d}_1.$$

8.2 BOUNDARY CONDITIONS OF TYPE EQUATIONS (4) AND (5)

The first of these (equations (4)) are of the form

$$p_k(L) - p_{k-1}(L) = h_k\,\sigma_k, \quad k \in I_c.$$

The functions $p_0(y)$ and $p_{N+1}(y)$ are defined to be identically zero for all y. Thus, for $k = N+1$ (i.e., the $N + 1$th ply element is cracked), we have

$$-p_N(L) = h_N\,\sigma_N$$

and for $k = 1$ (i.e., the first ply element is cracked), we have

$$p_1(L) = h_1\,\sigma_1.$$

Now, from equation (25), we see that

$$p_k(L) - p_{k-1}(L) = \sum_{i=1}^{3N} A_i^* \left(\mu_{i,2N-k} - \mu_{i,2N-k+1}\right)\left(1 + e^{-2\lambda_i L}\right)$$

and so, if we set up the matrix $\boldsymbol{\mu}_2$ defined by

$$\mu_2^{i,l} = \mu_{l,2N-I_c(i)} - \mu_{l,2N-I_c(i)+1}, \quad i = 1,2,\ldots c, \quad l = 1,2,\ldots,3N,$$

where $I_c(i)$ is the ith element of the set I_c, i.e., the index of the ith cracked ply, the μ values for functions p_1, p_{N+1} are set to zero, and the right hand side vector $\mathbf{d}_2$ given by

$$\mathbf{d}_2 = \begin{pmatrix} h_{I_c(1)}\,\sigma_{I_c(1)} \\ h_{I_c(2)}\,\sigma_{I_c(2)} \\ \vdots \\ h_{I_c(c)}\,\sigma_{I_c(c)} \end{pmatrix}$$

then the subsystem for these boundary conditions may be expressed as

$$\boldsymbol{\mu}_2\,\mathbf{E}^+\,\mathbf{A}^* = \mathbf{d}_2.$$

Equations (5) are of a similar type to (4), except they involve the $q(y)$ functions instead of the $p(y)$ functions. Suppose that $\boldsymbol{\mu}_3$ is defined by

$$\mu_3^{i,l} = \mu_{l,N-I_c(i)} - \mu_{l,N-I_c(i)+1}, \quad i = 1,2,\ldots u, \quad l = 1,2,\ldots,3N$$

and $\mathbf{d}_3$ is

$$\mathbf{d}_3 = \begin{matrix} h_{I_c(1)}\,\tau_{I_c(1)} \\ h_{I_c(2)}\,\tau_{I_c(2)} \\ \vdots \\ h_{I_c(c)}\,\tau_{I_c(c)} \end{matrix}$$

then these equations may be expressed as

$$\boldsymbol{\mu}_3\,\mathbf{E}^+\,\mathbf{A}^* = \mathbf{d}_3.$$

If we define the following matrices:

$$\boldsymbol{\Omega}^2 = \boldsymbol{\mu}_2\,\mathbf{E}^+,$$
$$\boldsymbol{\Omega}^3 = \boldsymbol{\mu}_3\,\mathbf{E}^+$$

then the system of boundary conditions (4) may be written

$$\boldsymbol{\Omega}^2\,\mathbf{A}^* = \mathbf{d}_2$$

and that of boundary conditions (5) as

$$\boldsymbol{\Omega}^3\,\mathbf{A}^* = \mathbf{d}_3$$

8.3 BOUNDARY CONDITIONS OF TYPE EQUATIONS (6) AND (7)

From equation (26), we have, for example

$$p^*_{k-1}(L) - p^*_k(L) = \sum_{i=1}^{3N} \frac{A^*_i}{\lambda_i}\left(1 - e^{-2\lambda_i L}\right)(\mu_{i,2N-k+1} - \mu_{i,2N-k}),$$

where $k \in I_u \setminus j$. Let us make the definition

$$\overline{\Lambda} = \begin{matrix} \lambda_1^{-1} & 0 & & 0 \\ 0 & \lambda_2^{-1} & & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & & 0 & \lambda_{3N}^{-1} \end{matrix}$$

and define μ_4 as follows:

$$\begin{aligned} &= \frac{\overline{g}^i_{22}}{h_i L}(\mu_{l,2N-i+1} - \mu_{l,2N-i}) - \frac{\overline{g}^j_{22}}{h_j L}(\mu_{l,2N-j+1} - \mu_{l,2N-j}) \\ &+ \frac{\overline{g}^i_{24}}{h_i L}(\mu_{l,N-i-1} - \mu_{l,N-i}) - \frac{\overline{g}^j_{24}}{h_j L}(\mu_{l,N-j-1} - \mu_{l,N-j}), \quad i \in I_u \setminus j \end{aligned} \tag{27}$$

Then the subsystem for equations (6) may be expressed as

$$\boldsymbol{\mu}_4 \overline{\Lambda} \mathbf{E}^- \mathbf{A}^* = \mathbf{d}_4,$$

where $\mathbf{d}_4$ is the null vector of length $u - 1$

If we define the matrix μ_5 as

$$\begin{aligned} &= \frac{\overline{g}^i_{24}}{h_i L}(\mu_{l,2N-i+1} - \mu_{l,2N-i}) - \frac{\overline{g}^j_{24}}{h_j L}(\mu_{l,2N-j+1} - \mu_{l,2N-j}) \\ &+ \frac{\overline{g}^i_{44}}{h_i L}(\mu_{l,N-i-1} - \mu_{l,N-i}) - \frac{\overline{g}^j_{44}}{h_j L}(\mu_{l,N-j-1} - \mu_{l,N-j}), \quad i \in I_u \setminus j, \end{aligned} \tag{28}$$

then the subsystem for equations (7) may be written

$$\boldsymbol{\mu}_5 \overline{\Lambda} \mathbf{E}^- \mathbf{A}^* = \mathbf{d}_5,$$

where $\mathbf{d}_5$ is the null vector of length $u - 1$. Define the following matrices:

$$\Omega^4 = \boldsymbol{\mu}_4 \overline{\Lambda} \mathbf{E}^-,$$

$$\Omega^5 = \boldsymbol{\mu}_5 \overline{\Lambda} \mathbf{E}^-.$$

Then the system of boundary conditions (6) may be expressed as

$$\Omega^4 \mathbf{A}^* = \mathbf{d}_4$$

and the system of boundary conditions (7) as

$$\Omega^5 \mathbf{A}^* = \mathbf{d}_5$$

8.4 ASSEMBLY OF COMPLETE SET OF BOUNDARY CONDITIONS

Having established the subsystem matrices and right hand side vectors for the sets of different types of boundary conditions, we assemble these into global system matrices thus:

$$\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Omega}^1 \\ \boldsymbol{\Omega}^2 \\ \boldsymbol{\Omega}^3 \\ \boldsymbol{\Omega}^4 \\ \boldsymbol{\Omega}^5 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \\ \mathbf{d}_4 \\ \mathbf{d}_5 \end{pmatrix}$$

The matrices $\boldsymbol{\Omega}^1, \boldsymbol{\Omega}^2, \boldsymbol{\Omega}^3$, $\boldsymbol{\Omega}^4$ and $\boldsymbol{\Omega}^5$ are of dimensions $N \times 3N, c \times 3N, c \times 3N, u-1 \times 3N$ and $u-1 \times 3N$ respectively. Recall that $c + u = N + 1$ and so the matrix $\boldsymbol{\Omega}$ is of dimensions $3N \times 3N$, as required.

We solve the $3N \times 3N$ system of linear algebraic equations

$$\boldsymbol{\Omega}\mathbf{A}^* = \mathbf{d}$$

for the unknown arbitrary constants A_i^*, $i = 1, 2, \ldots, 3N$ using the LINPACK[9] Fortran subroutine ZGESL.

9 THE FORM OF THE SOLUTION FUNCTIONS

For the data provided, some of the (independent) roots of the characteristic equation are real-valued, while the others have complex-conjugate values. Correspondingly, if $\lambda_i = \overline{\lambda_j}$ for any integers i and j, then the coefficients A_i^* and A_j^* also have complex-conjugate values, yielding real values for $p_k(y), q_k(y), k = 1, 2, \ldots, N$.

10 EXAMPLE

The above method of solution has been applied to a particular case. The software that is used to perform the tasks described in this report is embedded within software for the overall general symmetric multiple-ply laminate model. The differential equation-solving software is in practice executed several times (once for each of the crack sites considered) in a given analysis. For the purposes of this report, however, we provide details (data, results) corresponding to a single crack site.

10.1 DATA

In the case considered here, one half of the symmetric laminate consisted of just two plies. The first ply was damaged (cracked), was 0.125mm thick and was oriented at 90°. The second ply was undamaged, 0.25mm thick and was oriented at 55°. Each ply was modelled using three ply elements. In this way, there were five ply element interfaces ($N = 5$) and so there were ten simultaneous differential equations to be solved subject to thirty boundary conditions (fifteen after taking symmetry about $y = 0$ into account). The half-length of the plies was taken to be $L = 0.5$mm.

Details of the matrices required as data for the differential equations and boundary conditions are extensive and may be found in Appendix A.

RESULTS

For the above-mentioned problem, solutions were sought at $y = 0$ and $y = L$. The solution functions were found to take the following values here:

Function	y = 0	y = L
p_1	0.0033041732	0.0322649900
p_2	0.0061558718	0.0645299799
p_3	0.0081651505	0.0967949699
p_4	0.0126244193	0.0157897331
p_5	0.0092656051	0.0065248288
q_1	0.0006224284	0.0124377553
q_2	0.0012277147	0.0248755106
q_3	0.0017990508	0.0373132659
q_4	-0.0020960741	0.0155904104
q_5	-0.0024262970	0.0093262107

RESIDUALS

To obtain a measure of the accuracy of the solution obtained, we substituted the solution values back into the differential equations and corresponding boundary conditions. Although the method used was semi-analytical, it was implemented on a finite-precision computer. Clearly, if the *exact* solution $p-$ and $q-$values were substituted into the left hand side of the first equation of (1), then a value of zero would be given, since this equation is homogeneous. However, with inexact values, the value obtained is nonzero and (its absolute value) is called the *residual* for this equation. In this section, we quote the residuals obtained for each of the differential equations and boundary conditions.

10.3.1 Residuals for equations (1) and (2)

There were five differential equations of fourth order (1) and five of second order (2). The residuals for each of these are tabulated below.

Equation number	4th order equation	2nd order equation
1	3.05×10^{-15}	6.51×10^{-17}
2	4.64×10^{-12}	1.74×10^{-14}
3	2.84×10^{-12}	7.09×10^{-15}
4	1.36×10^{-13}	4.60×10^{-15}
5	2.45×10^{-13}	8.66×10^{-16}

10.3.2 Residuals for boundary conditions (3)

Here, we present the residuals for the five zero-derivative boundary conditions, namely equations (3).

Equation number	
1	4.07×10^{-14}
2	4.41×10^{-14}
3	2.26×10^{-15}
4	3.46×10^{-15}
5	7.77×10^{-16}

10.3.3 Residuals for boundary conditions (4) and (5)

These two types of boundary conditions require a simple rearrangement in order to present them in homogeneous form, namely

$$p_i(L) - p_{i-1}(L) - h_i \sigma_i = 0,$$

$$q_i(L) - q_{i-1}(L) - h_i \tau_i = 0.$$

There were three cracked ply elements representing the 90° ply. The resulting residuals are tabulated below:

Equation number	Residual for (4)	Residual for (5)
1	4.30×10^{-16}	5.55×10^{-17}
2	1.50×10^{-15}	4.51×10^{-17}
3	7.22×10^{-16}	7.98×10^{-17}

10.3.4 Residuals for boundary conditions (6) and (7)

The nearest uncracked ply element to the centre plane of the laminate was ply element number 4 and so each of equations (6) and (7) were required to be satisfied for ply elements 5 and 6. The residuals for these boundary conditions are shown below.

Equation number	Residual for (6)	Residual for (7)
1	1.04×10^{-17}	1.32×10^{-17}
2	3.01×10^{-17}	3.84×10^{-17}

Many more problems have been solved using this analysis and methodology. Applications involving up to forty ply elements have been shown to give good solutions.

11 CONCLUSIONS

A new method for modelling general symmetric multiple-ply laminates has been developed by McCartney [1]. This method is a discretisation procedure which involves the representation of composite materials by a set of perfectly bonded ply elements. In special cases where the laminate is cross-ply (i.e. all the plies are either 0° or 90°), the method described in this report are not applicable, since such cases do not correspond to the coupled differential equation systems (1) and (2). However, the model appropriate to the cross-ply problem is described in (McCartney and Saunderson, 1995)[2] and the method of solution of the decoupled differential equations is found in (Hannaby, 1993)[4].

The analysis of the general symmetric multiple-ply laminate gives rise to systems of ordinary differential equations and appropriate boundary conditions, the solution of which is considered in this report. It has been demonstrated, by using the characteristic equation of the system, that eigensolutions may be found and that the computation of unknown coefficients may be carried out in a stable manner. This is achieved by giving careful consideration to the treatment of potentially large exponents arising from the characteristic equation solution. Moreover, we have shown how the size of the system of linear algebraic equations which is consistent with the prescribed boundary conditions may be considerably reduced.

We have demonstrated, by tabulating residuals, that the solution values obtained satisfy the underlying differential equations and boundary conditions exceedingly well. The residuals were found to be typically of the order of 10^{-12} times smaller than the $p-$ and $q-$function values, which corresponds to very acceptable accuracy.

12 ACKNOWLEDGEMENTS

The analytical approach used to formulate the coefficients of the systems of differential equations and corresponding boundary conditions, and the work described in this report, were carried out as part of the NPL Strategic Research programme. I would like to thank Dr L N McCartney for carrying out the work of the first part of this project, namely the analysis of the general symmetric multiple-ply laminate method and for his valuable comments on earlier drafts of this report.

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A INPUT DATA

We provide here the matrices and vectors required for equations (1) to (7):

F

-.53331E-04	-.47698E-04	-.40411E-04	-.29390E-04	-.14695E-04
-.10103E-03	-.93742E-04	-.80761E-04	-.58780E-04	-.29390E-04
-.22755E-03	-.21653E-03	-.19455E-03	-.14547E-03	-.72967E-04
-.39092E-03	-.37623E-03	-.34684E-03	-.27433E-03	-.14344E-03
-.44712E-03	-.43242E-03	-.40303E-03	-.33007E-03	-.18663E-03

G:

.70536E-02	.14107E-02	.00000E+00	.00000E+00	.00000E+00
.84643E-02	.70536E-02	.14107E-02	.00000E+00	.00000E+00
.88542E-02	.88542E-02	.74435E-02	.59863E-03	-.94205E-03
.13954E-01	.13954E-01	.13954E-01	.12414E-01	.24827E-02
.11472E-01	.11472E-01	.11472E-01	.12414E-01	.99309E-02

H:

-.24377E+01	.24377E+01	.00000E+00	.00000E+00	.00000E+00
.00000E+00	-.24377E+01	.24377E+01	.00000E+00	.00000E+00
.00000E+00	.00000E+00	-.24377E+01	.92854E+00	.00000E+00
.00000E+00	.00000E+00	.00000E+00	-.92854E+00	.92854E+00
-.92854E+00	-.92854E+00	-.92854E+00	-.92854E+00	-.18571E+01

g:

.19754E-18	.39508E-19	.00000E+00	.00000E+00	.00000E+00
.23705E-18	.19754E-18	.39508E-19	.00000E+00	.00000E+00
.67553E-02	.67553E-02	.67553E-02	.45610E-02	.19794E-02
.36138E-02	.36138E-02	.36138E-02	.30115E-02	.60230E-03
.30115E-02	.30115E-02	.30115E-02	.30115E-02	.24092E-02

h:

-.13654E-15	.13654E-15	.00000E+00	.00000E+00	.00000E+00
.00000E+00	-.13654E-15	.13654E-15	.00000E+00	.00000E+00
.00000E+00	.00000E+00	-.13654E-15	.59940E+00	.00000E+00
.00000E+00	.00000E+00	.00000E+00	-.59940E+00	.59940E+00
-.59940E+00	-.59940E+00	-.59940E+00	-.59940E+00	-.11988E+01

A:

.19754E-18	.39508E-19	.00000E+00	.00000E+00	.00000E+00
.23705E-18	.19754E-18	.39508E-19	.00000E+00	.00000E+00
.27966E-02	.27966E-02	.27966E-02	.60230E-03	.00000E+00
.55931E-02	.55931E-02	.55931E-02	.30115E-02	.60230E-03
.49909E-02	.49909E-02	.49909E-02	.30115E-02	.24092E-02

B:

-.13654E-15	.13654E-15	.00000E+00	.00000E+00	.00000E+00
.00000E+00	-.13654E-15	.13654E-15	.00000E+00	.00000E+00
.00000E+00	.00000E+00	-.13654E-15	.59940E+00	.00000E+00
.00000E+00	.00000E+00	.00000E+00	-.59940E+00	.59940E+00
-.59940E+00	-.59940E+00	-.59940E+00	-.59940E+00	-.11988E+01

a

.53637E-02	.10727E-02	.00000E+00	.00000E+00	.00000E+00
.64364E-02	.53637E-02	.10727E-02	.00000E+00	.00000E+00
.11613E-01	.11613E-01	.10540E-01	.27982E-02	.00000E+00
.16789E-01	.16789E-01	.16789E-01	.13991E-01	.27982E-02
.13991E-01	.13991E-01	.13991E-01	.13991E-01	.11193E-01

b:

-.37074E+01	.37074E+01	.00000E+00	.00000E+00	.00000E+00
.00000E+00	-.37074E+01	.37074E+01	.00000E+00	.00000E+00
.00000E+00	.00000E+00	-.37074E+01	.76174E+00	.00000E+00
.00000E+00	.00000E+00	.00000E+00	-.76174E+00	.76174E+00
-.76174E+00	-.76174E+00	-.76174E+00	-.76174E+00	-.15235E+01

σ:

0.77436	0.77436	0.77436	1.11282	1.11282	1.11282

τ:

0.29851	0.29851	0.29851	-0.14925	-0.14925	-0.14925

Ply element thickness:

0.00417	0.00417	0.00417	0.08333	0.08333	0.08333

$\overline{g}_{22}$

0.10157	0.10157	0.10157	0.07738	0.07738	0.07738

$\overline{g}_{24}$

0.00000	0.00000	0.00000	0.04995	0.04995	0.04995

$\overline{g}_{44}$

0.15447	0.15447	0.15447	0.06348	0.06348	0.06348