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ABSTRACT

Stress transfer in a multiphase unidirectional composite has been modelled by a multiple concentric cylinder technique, whereby the fibre, surrounding matrix and surrounding composite are represented by a series of concentric cylinders. The cylinders are assumed to be perfectly bonded together and the case where loading causes the fibre to crack has been considered. This model generates a system of simultaneous fourth order ordinary differential equations with appropriate boundary conditions. The purpose of this report is to describe a semi-analytic method for solving these differential equations.

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CONTENTS

1 INTRODUCTION

This report describes a semi-analytical method for the solution of a system of simultaneous fourth order ordinary differential equations arising from the analysis of the stress transfer mechanics of a unidirectional multiphase composite. Multiphase composites are encountered when coated fibres are incorporated into composite materials. The problem considered is that of a three-phase composite which is subjected to uniaxial tension.

In (McCartney, 1993), a multiple concentric cylinder model of stress transfer in unidirectional composites is developed whereby the fibre and matrix are represented by a series of concentric cylinders. The axial loading of a composite having a broken fibre gives rise to axial stress transfer between the cylinders through distributed shear stresses on the interfaces between neighbouring cylinders. It is assumed that the concentric cylinders are perfectly bonded together. The general model allows each cylinder to be made of a different transverse isotropic material. However, in the application of the model to the three-phase composite problem, it is ensured that the cylinders representing the fibre possess one set of mechanical properties, those representing the matrix have a different set and those representing the surrounding composite have another. The loading is assumed to lead to the cracking of the fibre into fragments of equal length and one such section of length $2L$ is modelled. The model develops sets of recurrence relations which are used to generate the system of ordinary differential equations whose solution is considered here.

In Section 2, we introduce the differential equations and boundary conditions which describe the problem. The auxiliary/characteristic equation, which determines eigensolutions of the equations, is explained in Section 3. Once this equation has been solved, the solution functions, $C_i(z)$, are known to within a set of arbitrary constants. The different types of boundary conditions are then expressed (Section 4) in the form of a square set of simultaneous linear algebraic equations. The unknowns of these equations are the arbitrary constants which fully determine the $C_i(z)$ functions.

In Section 5, we show that only some of the arbitrary constants are independent and explain that we may consequently reduce the order of the system of equations that needs to be solved. A possible source of numerical error is identified in Section 6. This is where some of the eigensolutions possess large exponents. We prescribe a means of overcoming this problem. We indicate that the solution functions $C_i(z)$ must be real in Section 7 and provide an example and corresponding results in Section 8.

2 THE DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

Following McCartney (1993), the system of N differential equations for $N+1$ cylinders (N interfaces, $r = r_i, i = 1, \dots, N$) is written

$$
\begin{aligned}
&F_{11}C_1''''(z) + G_{11}C_1''(z) + H_{11}C_1(z) + F_{21}C_2''''(z) + G_{21}C_2''(z) + H_{21}C_2(z) + \\
&\dots + F_{N1}C_N''''(z) + G_{N1}C_N''(z) + H_{N1}C_N(z) = 0, \\
&F_{12}C_1''''(z) + G_{12}C_1''(z) + H_{12}C_1(z) + F_{22}C_2''''(z) + G_{22}C_2''(z) + H_{22}C_2(z) + \\
&\dots + F_{N2}C_N''''(z) + G_{N2}C_N''(z) + H_{N2}C_N(z) = 0, \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&F_{1N}C_1''''(z) + G_{1N}C_1''(z) + H_{1N}C_1(z) + F_{2N}C_2''''(z) + G_{2N}C_2''(z) + H_{2N}C_2(z) + \\
&\dots + F_{NN}C_N''''(z) + G_{NN}C_N''(z) + H_{NN}C_N(z) = 0,
\end{aligned} \tag{1}
$$

for $z \in [-L, L]$, where the F_{ij}, G_{ij}, H_{ij} are constant coefficients (derived from recurrence relations). The $C_i(z)$ are the unknowns to be determined and their derivatives are identically equal to the shear stress components, i.e.

$$\sigma_{rz}(r_i, z) \equiv C_i'(z), \quad i = 1, \dots, N.$$

The $4 \times N$ boundary conditions are written

$$C_1(\pm L) = \frac{1}{2} r_1 \sigma_1 = \alpha_1, \tag{2}$$

$$C_i(\pm L) = \frac{r_{i-1}}{r_i} C_{i-1}(\pm L) + \frac{r_{i-1}}{2\omega_i} \sigma_i = \alpha_i, \quad i = 2, \dots, N_f, \tag{3}$$

$$C_i'(\pm L) = 0, \quad i = 1, \dots, N_f - 1, N_f + 1, \dots, N, \tag{4}$$

$$\Delta u_i'(\pm L) = 0, \quad i = N_f + 1, \dots, N, \tag{5}$$

$$\Delta u_{N_f}'(\pm L) = \frac{C_{N_f}'(\pm L)}{\mu_A^{N_f+1}}, \tag{6}$$

where $\mu_A^{N_f+1}$ and σ_i are prescribed constants and N_f is the number of cylinders which represent the fibre. The ω_i are given by

$$\omega_i = \frac{r_i r_{i-1}}{r_i^2 - r_{i-1}^2}$$

and we shall assume for applications that L is at least five times as large as r_1.

Boundary conditions (2), (3) and (4) are straightforward function value and derivative boundary conditions whereas boundary conditions (5) and (6) require more explanation. The function $\Delta u_j'(z)$ is given by

$$\Delta u_j'(z) = \sum_{i=1}^{N} D_{ij} C_i'''(z) + \sum_{i=1}^{N} E_{ij} C_i'(z), \tag{7}$$

where D_{ij}, E_{ij} are again known constants.

3 THE AUXILIARY/CHARACTERISTIC EQUATION

The characteristic equation for system (1) is

$$|M_{ij}(\lambda)| = 0, \tag{8}$$

where

$$M_{ij}(\lambda) = \lambda^4 F_{ij} + \lambda^2 G_{ij} + H_{ij}. \tag{9}$$

The left-hand side of equation (8) is a $4N^{\text{th}}$ degree polynomial in λ and if $\lambda_i, i = 1, \dots, 4N$ are its roots, then we may write, without loss of generality,

$$C_N(z) = \sum_{i=1}^{4N} A_i e^{\lambda_i z}. \tag{10}$$

Rewriting equations (1) as

$$\begin{aligned}
L_{11}(D)C_1(z) + L_{21}(D)C_2(z) + \dots + L_{N1}(D)C_N(z) &= 0, \\
L_{12}(D)C_1(z) + L_{22}(D)C_2(z) + \dots + L_{N2}(D)C_N(z) &= 0, \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
L_{1N}(D)C_1(z) + L_{2N}(D)C_2(z) + \dots + L_{NN}(D)C_N(z) &= 0,
\end{aligned} \tag{11}$$

where

$$L_{ij}(D) \equiv F_{ij}D^4 + G_{ij}D^2 + H_{ij},$$

and computing the LU factorisation of the corresponding matrix operator **L**, we obtain an upper triangular matrix U, which we choose to rename as $\overline{L}$ to emphasise that it is a matrix of linear differential operators. In terms of this upper triangular matrix operator, the system of differential equations (1) can be written as

$$\begin{bmatrix} \overline{L}_{11} & \overline{L}_{21} & \overline{L}_{31} & \dots & \overline{L}_{N1} \\ 0 & \overline{L}_{22} & \overline{L}_{32} & \dots & \overline{L}_{N2} \\ 0 & 0 & \overline{L}_{33} & \dots & \overline{L}_{N3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \overline{L}_{NN} \end{bmatrix} \begin{pmatrix} C_1(z) \\ C_2(z) \\ C_3(z) \\ \vdots \\ C_N(z) \end{pmatrix} = \mathbf{0}.$$

The last three equations of this system are:

$$\overline{L}_{NN}C_N(z) = 0, \tag{12}$$

$$\overline{L}_{N-1,N-1}C_{N-1}(z) + \overline{L}_{N-1,N}C_N(z) = 0 \tag{13}$$

and

$$\overline{L}_{N-2,N-2}C_{N-2}(z) + \overline{L}_{N-2,N-1}C_{N-1}(z) + \overline{L}_{N-2,N}C_N(z) = 0, \tag{14}$$

in reverse order. The eigensolutions $e^{\lambda_i z}$, which have been derived from equation (8) for $C_N(z)$, are also the eigensolutions for the other $C_i(z)$, $i = 1, \dots, N-1$.

We may rewrite equation (13) as

$$\overline{L}_{N-1,N-1}C_{N-1}(z) = -\overline{L}_{N-1,N}C_N(z).$$

Using the *D-operator method* (Chorlton, 1965), we see that if $C_N(z)$ is of the form (10), then $C_{N-1}(z)$ must be given by

$$C_{N-1}(z) = \sum_{i=1}^{4N} A_i\mu_{i1}e^{\lambda_i z},$$

where

$$\mu_{i1} = -\frac{\overline{M}_{N-1,N}}{\overline{M}_{N-1,N-1}} \tag{15}$$

and $\overline{M}_{ij}$ is the upper triangular factor of M_{ij} given by equation (9). Moreover, equation (14) may be rewritten as

$$\overline{L}_{N-2,N-2}C_{N-2}(z) = -\overline{L}_{N-2,N-1}C_{N-1}(z) - \overline{L}_{N-2,N}C_N(z),$$

the D-operator method giving

$$C_{N-2}(z) = \sum_{i=1}^{4N} A_i\mu_{i2}e^{\lambda_i z},$$

where

$$\mu_{i2} = -\frac{\overline{M}_{N-2,N-1}\mu_{i1} + \overline{M}_{N-2,N}}{\overline{M}_{N-2,N-2}}. \tag{16}$$

This process of backsubstitution is followed until all the $C_i(z)$ have been found. If we make the definition $\mu_{i,0} = 1$, $i = 1, \dots, 4N$, then we may write generally

$$C_k(z) = \sum_{i=1}^{4N} A_i\,\mu_{i,N-k}\,e^{\lambda_i z}, \quad k = 1, \dots, N. \tag{17}$$

4 INCLUSION OF THE BOUNDARY CONDITIONS

The $4N$ boundary conditions are given by (2) (2 conditions), (3) ($2(N_f - 1)$ conditions), (4) ($2(N-1)$ conditions), (5) ($2(N-N_f)$ conditions) and (6) (2 conditions). We shall consider how these boundary conditions are incorporated into a square system of linear algebraic equations in the next subsections. We consider boundary conditions of type (2) or (3) first, boundary conditions of type (4) second and boundary conditions of type (5), (6) last.

4.1 BOUNDARY CONDITIONS OF TYPE (2) OR (3)

These boundary conditions are of the form

$$C_i(\pm L) = \alpha_i, \quad i = 1, \ldots, N_f.$$

If we consider a typical solution function

$$C_k(z) = \sum_{i=1}^{4N} A_i \mu_{i,N-k}\, e^{\lambda_i z},$$

we see that the corresponding pair of boundary conditions may be written

$$\begin{bmatrix} \mu_{1,N-k}e^{\lambda_1 L} & \mu_{2,N-k}e^{\lambda_2 L} & \ldots & \mu_{4N,N-k}e^{\lambda_{4N} L} \\ \mu_{1,N-k}e^{-\lambda_1 L} & \mu_{2,N-k}e^{-\lambda_2 L} & \ldots & \mu_{4N,N-k}e^{-\lambda_{4N} L} \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{4N} \end{pmatrix} = \begin{pmatrix} \alpha_k \\ \alpha_k \end{pmatrix}.$$

4.2 BOUNDARY CONDITIONS OF TYPE (4)

These boundary conditions are of the form

$$C_i'(\pm L) = 0, \quad i = 1, \ldots, N_f - 1, N_f + 1, \ldots, N.$$

The pair of conditions for function $C_k(z)$ may therefore be written

$$\begin{bmatrix} \mu_{1,N-k}\lambda_1 e^{\lambda_1 L} & \mu_{2,N-k}\lambda_2 e^{\lambda_2 L} & \ldots & \mu_{4N,N-k}\lambda_{4N} e^{\lambda_{4N} L} \\ -\mu_{1,N-k}\lambda_1 e^{-\lambda_1 L} & -\mu_{2,N-k}\lambda_2 e^{-\lambda_2 L} & \ldots & -\mu_{4N,N-k}\lambda_{4N} e^{-\lambda_{4N} L} \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{4N} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

4.3 BOUNDARY CONDITIONS OF TYPE (5) OR (6)

These are far more involved than the other types of boundary condition. Consider first the homogeneous boundary conditions of type (5). They are specified by

$$\Delta u_i'(\pm L) = 0, \quad i = N_f + 1, \ldots, N.$$

From (7), we see that these equations may be written

$$\sum_{j=1}^{N} D_{ij} C_j'''(L) + \sum_{j=1}^{N} E_{ij} C_j'(L) = 0,$$

$$\sum_{j=1}^{N} D_{ij} C_j'''(-L) + \sum_{j=1}^{N} E_{ij} C_j'(-L) = 0,$$

for $i = N_f + 1, \ldots, N$. Recalling that the general function $C_k(z)$ is of the form (17), we see that

$$C_k'''(z) = \sum_{i=1}^{4N} A_i \, \mu_{i,N-k} \, \lambda_i^3 \, e^{\lambda_i z},$$

and

$$C_k'(z) = \sum_{i=1}^{4N} A_i \, \mu_{i,N-k} \, \lambda_i \, e^{\lambda_i z}.$$

Thus for boundary condition i of this type at $z = L$, we have

$$\sum_{j=1}^{N} D_{ij} \sum_{l=1}^{4N} A_l \, \mu_{l,N-j} \, \lambda_l^3 \, e^{\lambda_l L} + \sum_{j=1}^{N} E_{ij} \sum_{l=1}^{4N} A_l \, \mu_{l,N-j} \, \lambda_l \, e^{\lambda_l L} = 0.$$

If we collect the terms involving coefficient A_s in this equation we have

$$T_{i,s}^{+} = \sum_{j=1}^{N} D_{ij} \, \mu_{s,N-j} \, \lambda_s^3 \, e^{\lambda_s L} + \sum_{j=1}^{N} E_{ij} \, \mu_{s,N-j} \, \lambda_s \, e^{\lambda_s L}.$$

For the inhomogeneous boundary condition of type (6), we have

$$\Delta u_{N_f}'(\pm L) = \frac{C_{N_f}'(\pm L)}{\mu_A^{N_f+1}}$$

and so the terms involving A_s for this equation at $z = L$ are given by

$$R_{N_f,s}^{+} = \sum_{j=1}^{N} D_{N_f,j} \, \mu_{s,N-j} \, \lambda_s^3 \, e^{\lambda_s L} + \sum_{j=1}^{N} E_{N_f,j} \, \mu_{s,N-j} \, \lambda_s \, e^{\lambda_s L} - \frac{1}{\mu_A^{N_f+1}} \, \mu_{s,N-N_f} \, \lambda_s \, e^{\lambda_s L}.$$

Let $T_{i,s}^{-}$ and $R_{N_f,s}^{-}$ be the corresponding terms when the boundary conditions are evaluated at $z = -L$. Then the complete set of boundary conditions may be represented by a square system of algebraic equations thus

$$
\begin{bmatrix}
\mu_{1,N-1}e^{\lambda_1 L} & \mu_{2,N-1}e^{\lambda_2 L} & \cdots \\
\mu_{1,N-1}e^{-\lambda_1 L} & \mu_{2,N-1}e^{-\lambda_2 L} & \cdots \\
\vdots & \vdots & \vdots \\
\mu_{1,N-N_f}e^{\lambda_1 L} & \mu_{2,N-N_f}e^{\lambda_2 L} & \cdots \\
\mu_{1,N-N_f}e^{-\lambda_1 L} & \mu_{2,N-N_f}e^{-\lambda_2 L} & \cdots \\
\mu_{1,N-1}\lambda_1 e^{\lambda_1 L} & \mu_{2,N-1}\lambda_2 e^{\lambda_2 L} & \cdots \\
-\mu_{1,N-1}\lambda_1 e^{-\lambda_1 L} & -\mu_{2,N-1}\lambda_2 e^{-\lambda_2 L} & \cdots \\
\vdots & \vdots & \vdots \\
\mu_{1,N-N_f+1}\lambda_1 e^{\lambda_1 L} & \mu_{2,N-N_f+1}\lambda_2 e^{\lambda_2 L} & \cdots \\
-\mu_{1,N-N_f+1}\lambda_1 e^{-\lambda_1 L} & -\mu_{2,N-N_f+1}\lambda_2 e^{-\lambda_2 L} & \cdots \\
\mu_{1,N-N_f-1}\lambda_1 e^{\lambda_1 L} & \mu_{2,N-N_f-1}\lambda_2 e^{\lambda_2 L} & \cdots \\
-\mu_{1,N-N_f-1}\lambda_1 e^{-\lambda_1 L} & -\mu_{2,N-N_f-1}\lambda_2 e^{-\lambda_2 L} & \cdots \\
\vdots & \vdots & \vdots \\
\lambda_1 e^{\lambda_1 L} & \lambda_2 e^{\lambda_2 L} & \cdots \\
-\lambda_1 e^{-\lambda_1 L} & -\lambda_2 e^{-\lambda_2 L} & \cdots \\
T^+_{N_f+1,1} & T^+_{N_f+1,2} & \cdots \\
T^-_{N_f+1,1} & T^-_{N_f+1,2} & \cdots \\
\vdots & \vdots & \vdots \\
T^+_{N,1} & T^+_{N,2} & \cdots \\
T^-_{N,1} & T^-_{N,2} & \cdots \\
R^+_{N_f,1} & R^+_{N_f,2} & \cdots \\
R^-_{N_f,1} & R^-_{N_f,2} & \cdots
\end{bmatrix}
\begin{pmatrix}
A_1 \\ A_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ A_{4N-1} \\ A_{4N}
\end{pmatrix}
=
\begin{pmatrix}
\alpha_1 \\ \alpha_1 \\ \vdots \\ \alpha_{N_f} \\ \alpha_{N_f} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{pmatrix}. \tag{18}
$$

5 REDUCTION OF THE EQUATIONS

Since the characteristic equation (8) is a $2N^{\text{th}}$ degree equation in λ^2, its $4N$ roots appear in positive and negative pairs ($\lambda_i = -\lambda_{i+2N}, i = 1, \ldots, 2N$). Consequently, only $2N$ of these are independent. Moreover, there are only $2N$ values of μ in each column given by expressions of the form (15) and (16) ($\mu_{ij} = \mu_{i+2N,j}, i = 1, \ldots, 2N$). From the symmetry of the problem (boundary conditions and formulation) it follows that only $2N$ of the A_i are independent, so that equations (18) reduce in number from $4N$ to $2N$. The coefficient of any particular $e^{\lambda z}$ must match that of the corresponding $e^{-\lambda z}$. Hence, we may set $A_i = A_{i+2N}, i = 1, \ldots, 2N$, whereupon the solution function $C_N(z)$ may be written

$$C_N(z) = A_1(e^{\lambda_1 z} + e^{-\lambda_1 z}) + A_2(e^{\lambda_2 z} + e^{-\lambda_2 z}) \ldots + A_{2N}(e^{\lambda_{2N} z} + e^{-\lambda_{2N} z}).$$

The system (18) can then be rewritten as

$$[\Omega](A) = (f), \tag{19}$$

where $[\Omega]$ is given by

$$\begin{bmatrix} \mu_{1,N-1}(e^{\lambda_1 L}+e^{-\lambda_1 L}) & \cdots & \mu_{2N,N-1}(e^{\lambda_{2N} L}+e^{-\lambda_{2N} L}) \\ \vdots & \vdots & \vdots \\ \mu_{1,N-N_f}(e^{\lambda_1 L}+e^{-\lambda_1 L}) & \cdots & \mu_{2N,N-N_f}(e^{\lambda_{2N} L}+e^{-\lambda_{2N} L}) \\ \mu_{1,N-1}\lambda_1(e^{\lambda_1 L}-e^{-\lambda_1 L}) & \cdots & \mu_{2N,N-1}\lambda_{2N}(e^{\lambda_{2N} L}-e^{-\lambda_{2N} L}) \\ \vdots & \vdots & \vdots \\ \mu_{1,N-N_f+1}\lambda_1(e^{\lambda_1 L}-e^{-\lambda_1 L}) & \cdots & \mu_{2N,N-N_f+1}\lambda_{2N}(e^{\lambda_{2N} L}-e^{-\lambda_{2N} L}) \\ \mu_{1,N-N_f-1}\lambda_1(e^{\lambda_1 L}-e^{-\lambda_1 L}) & \cdots & \mu_{2N,N-N_f-1}\lambda_{2N}(e^{\lambda_{2N} L}-e^{-\lambda_{2N} L}) \\ \vdots & \vdots & \vdots \\ \lambda_1(e^{\lambda_1 L}-e^{-\lambda_1 L}) & \cdots & \lambda_{2N}(e^{\lambda_{2N} L}-e^{-\lambda_{2N} L}) \\ \tilde{T}_{N_f+1,1} & \cdots & \tilde{T}_{N_f+1,2N} \\ \vdots & \vdots & \vdots \\ \tilde{T}_{N,1} & \cdots & \tilde{T}_{N,2N} \\ \tilde{R}_{N_f,1} & \cdots & \tilde{R}_{N_f,2N} \end{bmatrix},$$

$$(A) = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ \vdots \\ A_{2N} \end{pmatrix},$$

and

$$(f) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{N_f} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Here, $\tilde{T}_{i,s} = T^+_{i,s} + T^-_{i,s}$ and $\tilde{R}_{N_f,s} = R^+_{N_f,s} + R^-_{N_f,s}$.

6 TREATMENT OF LARGE EXPONENTS

Before the system of equations (19) can be solved, the matrix $[\Omega]$ must be computed. Here, a possible source of overflow or error can be identified by observing the arguments of the exponential functions in this matrix. If, for some λ_i, $i = 1, \ldots, 2N$, the product $\lambda_i L$ is too large, then the exponentiation may not be carried out.

This problem is overcome by dividing column i of $[\Omega]$ by the constant factor $e^{\lambda_i L}$ giving, for example, for element $(1, 1)$ of $[\Omega]$ the value

$$\mu_{1,N-1}\left(1+e^{-2\lambda_1 L}\right).$$

When $[\Omega]$ is modified in this way and the system is solved, we obtain as our solution the vector $(A^*) = [A_1^* A_2^* \ldots A_{2N}^*]$ instead of the vector (A). The relationship between A_i^* and A_i, for $i = 1, \ldots, 2N$, is clearly

$$A_i = e^{-\lambda_i L} A_i^*, \quad i = 1, \ldots, 2N. \tag{20}$$

However, for $i = 2N + 1, \ldots, 4N$, we have

$$\lambda_i = -\lambda_{i-2N},$$

whence, since we know that $A_i = A_{i+2N}$, $i = 1, \ldots, 2N$, we must have

$$A_i = e^{\lambda_i L} A_i^*, \qquad i = 2N + 1, \ldots, 4N.$$

The function $C_k(z)$, previously defined by equation (17), may therefore be written as

$$C_k(z) = \sum_{i=1}^{2N} A_i^* \, \mu_{i,N-k} \, e^{\lambda_i(z-L)} + \sum_{i=2N+1}^{4N} A_i^* \, \mu_{i,N-k} \, e^{\lambda_i(z+L)}.$$

This process removes the need to compute the A_i explicitly via equation (20). The latter practice would be very inaccurate for large values of $\lambda_i L$ since $e^{-\lambda_i L}$ could become so small that rounding errors reduce A_i to zero when it is not.

7 SOLUTION OF EQUATIONS

The characteristic equation (8) and the system of equations (19) (modified according to the discussion of Section 6) have been solved using MATLAB (Moler et al., 1988).

For the data provided, some of the (independent) roots of the characteristic equation are real, while the others have complex conjugate values. Correspondingly, if $\lambda_i = \overline{\lambda_j}$ for any integers i and j, the coefficients A_i^* and A_j^* also have complex conjugate values, yielding real values for $C_k(z)$, $k = 1, \ldots, N$.

8 EXAMPLE

The above method of solution has been applied to a particular case. The arrangement of the fibre, matrix and composite materials is shown in Figure 1. The innermost cylinder represents the fibre, the inner cylindrical hoop represents the matrix and the outer cylindrical hoop represents the composite. The length of the cylinders is 700μm and the radii of the fibre, the outer edge of the matrix and the outer edge of the composite are 3.5μm, 4.51848μm and 30μm respectively corresponding to a composite having a volume fraction of 0.6. The fibre, matrix and composite parts were discretised into 8, 4 and 9 cylinders respectively. The data required for the solution of equations (1) to (7) can be found in Appendix A. Note, however, that not all of the entries D_{ij} and E_{ij} are actually used for this problem. We only need rows $i = 1, \ldots, 20$ and columns $j = 8, \ldots, 20$ in this case.

8.1 RESULTS

We compare our results graphically with those of an axisymmetric finite element analysis carried out at the University of Bristol (Nedele, 1993), (Nedele and Wisnom, 1993) in Figures 2 to 9. The results plotted were all derived from the shear stress functions $C_i'(z)$ and in each figure the results from the concentric cylinder method are indicated by a solid line and the finite element results by a dotted line.

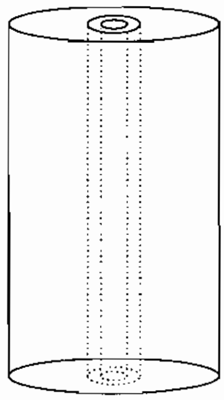

Figure 1 Illustration of fibre/matrix/composite arrangement.

8.2 DISCUSSION

Figure 2 shows the dependence of the axial stress σ_{zz} at the centre of the broken fibre as a function of the distance measured along the fibre. The fibre fracture is located at $z = 350\mu m$ and it is seen that the axial stress is zero there as required by the boundary conditions. The point $z = 0$ corresponds to the midpoint between two fibre breaks located at $z = \pm 350\mu m$. The results shown in this figure indicate that there is very good agreement between the predictions made using the concentric cylinder and finite element models.

The results of Figure 3 show the axial stress distribution at the location in the effective composite corresponding to the axes of the nearest neighbour fibres. These are used in the estimation of the stress concentration factor in the nearest neighbour fibre. This value is computed as 1.059 when using the concentric cylinder model and as 1.044 when using finite elements.

Figure 4 shows the shear stress σ_{rz} along the fibre/matrix interface as a function of distance along the interface. The shear stress has nonzero values only in regions where stress transfer between the fibre and matrix is occurring. Again, we see very good agreement between the two sets of results. Close to the fracture, however, both the analytical and finite element analysis predictions break down showing oscillatory behaviour. This is because neither method takes into account the singularity the exact solution must possess at that point. It should be clear however from Figure 4 that the analytical method is providing more reliable results near the fibre fracture than the finite element method.

Figure 5 shows the shear stress along the matrix/composite interface. In this case, the analytical method is predicting a higher maximum shear stress than the finite element method.

Figure 6 shows the radial (normal) stress σ_{rr} along the fibre/matrix interface. Figure 7 shows the corresponding distribution for the matrix/composite interface. The small discrepancy in these figures between the analytical and finite element results as $z \to 0$ is thought to arise because the finite element analysis was based on thermoelastic constants for the effective composite which differed slightly from those used for the analytical technique.

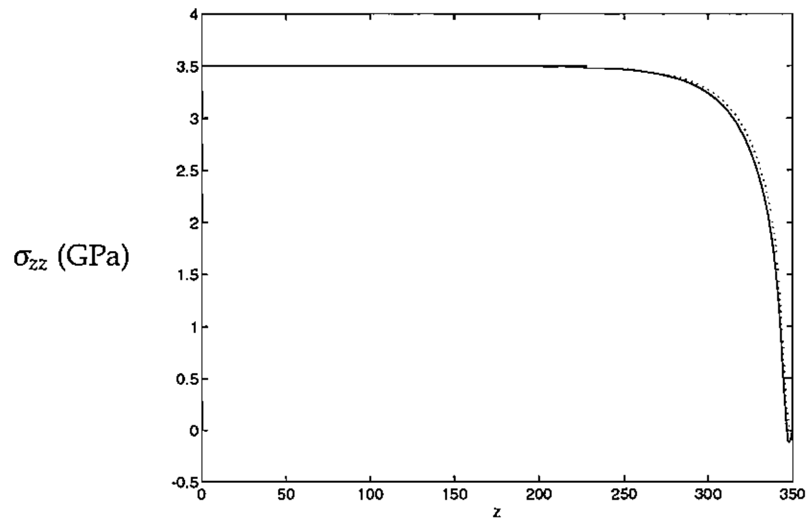

Figure 2 Axial stress along the centre of the broken fibre.

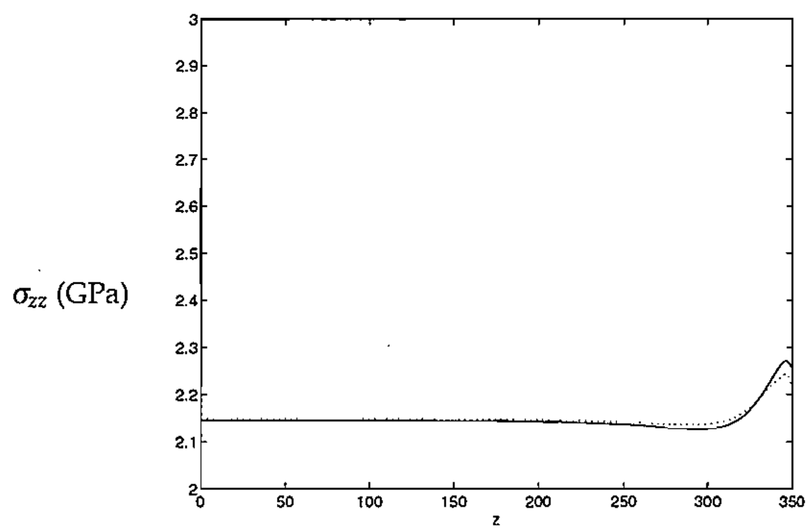

Figure 3 Axial stress at a distance corresponding to the location of the axes of neighbouring fibres.

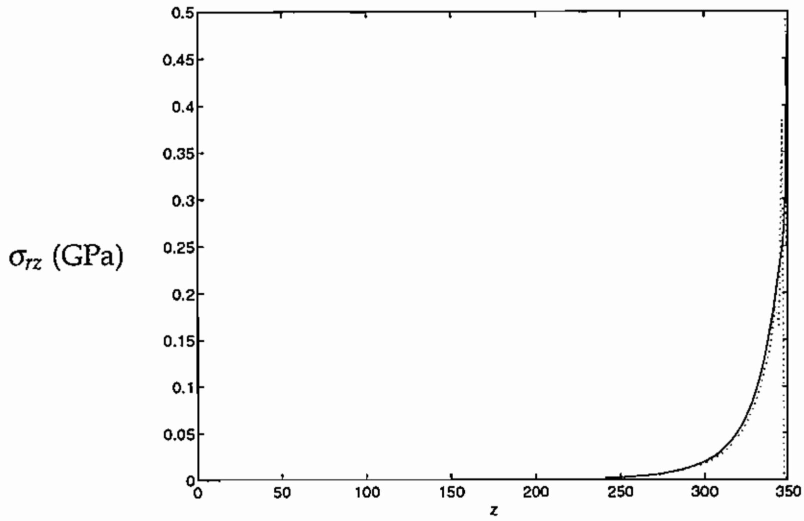

Figure 4 Shear stress along the fibre/matrix interface.

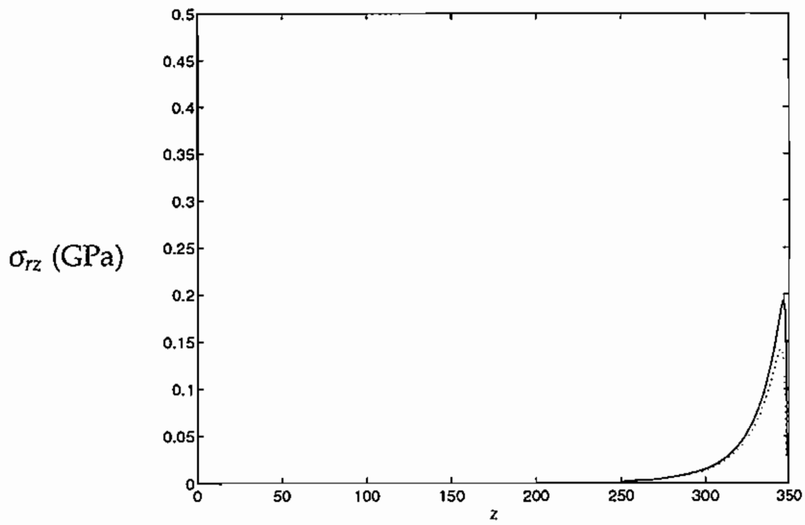

Figure 5 Shear stress along the matrix/composite interface.

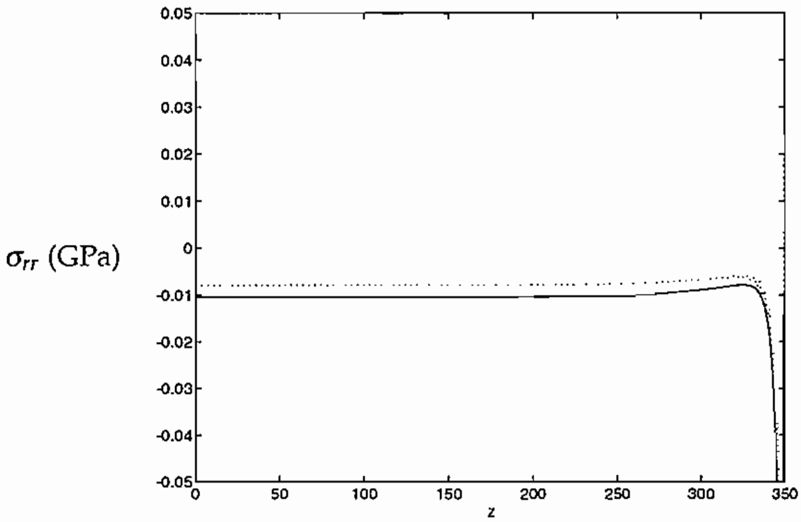

Figure 6 Radial stress along the fibre/matrix interface.

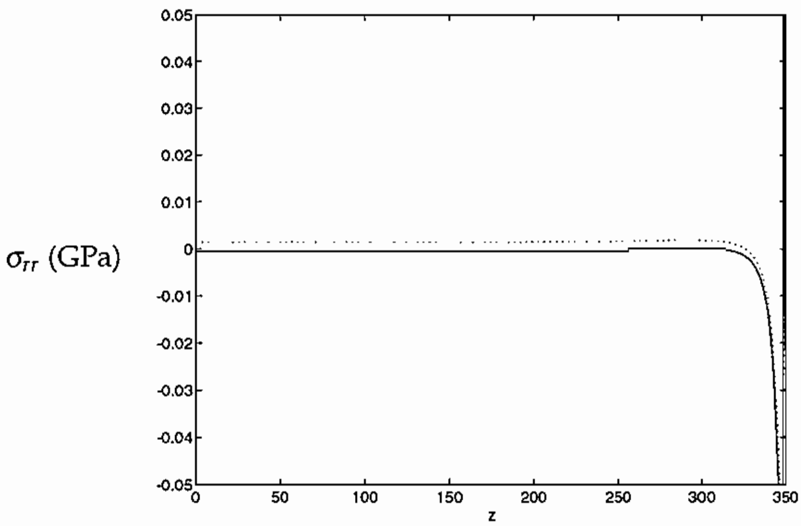

Figure 7 Radial stress along the matrix/composite interface.

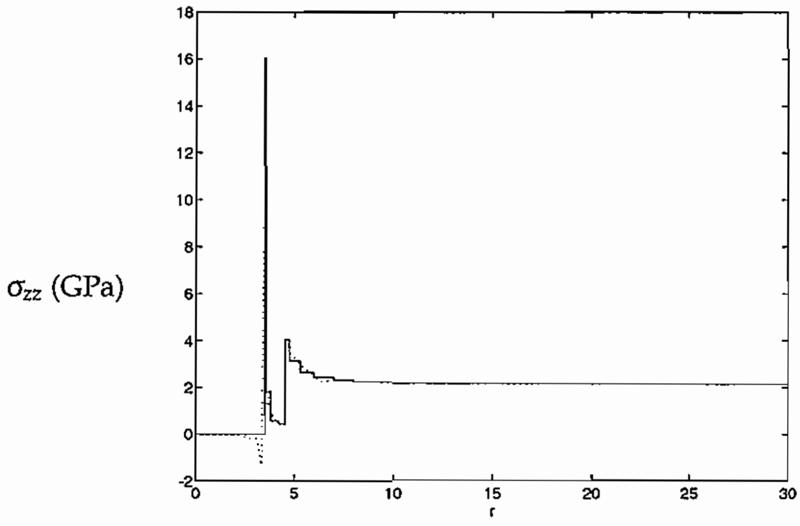

Figure 8 Axial stress along the crack plane (radial direction). The concentric cylinder solution is piecewise constant.

Figure 8 shows the dependence of the axial stress σ_{zz} in the plane of the cracked fibre on the radial distance r measured from the centre of the fibre. The analytical method solutions appear as a step function; the dotted line being the finite element solution. In the region representing the fibre ($0 \leq r \leq 3.5\mu$m) the analytical solution predicts a zero value for the axial stress thus satisfying the required boundary condition exactly. The finite element solution exhibits nonzero values in this region, violating the boundary condition. In the matrix material near the fibre/matrix interface, very high tensile stresses are encountered. The maximum value given by finite element analysis is 15.9 GPa and that given by the analytical method is 16.07 GPa thus showing very good agreement. The exact solution would predict an infinite axial stress at this point resulting from the singularity in the stress field. It can be seen that in the surrounding effective composite material the axial stress reduces from a maximum value (4.050 GPa for the analytical solution and 3.667 GPa for the finite element solution) at the matrix/composite interface to a constant value of 2.1424 GPa (same for both solutions). There is good agreement between the solutions due to each method although the analytical results are thought to be more reliable.

Figure 9 shows axial displacement in the plane of the fibre fracture relative to the uniform axial displacement of the matrix and effective composite material. Thus for the region $r \geq 3.5\mu$m both methods predict a zero value, satisfying the required displacement boundary condition. For the region $0 \leq r \leq 3.5\mu$m the analytical technique predicts a larger displacement difference than the finite element method. We note the very large displacement gradient which can be observed near the fibre/matrix interface, indicating again the effect of the singularity in the stress field.

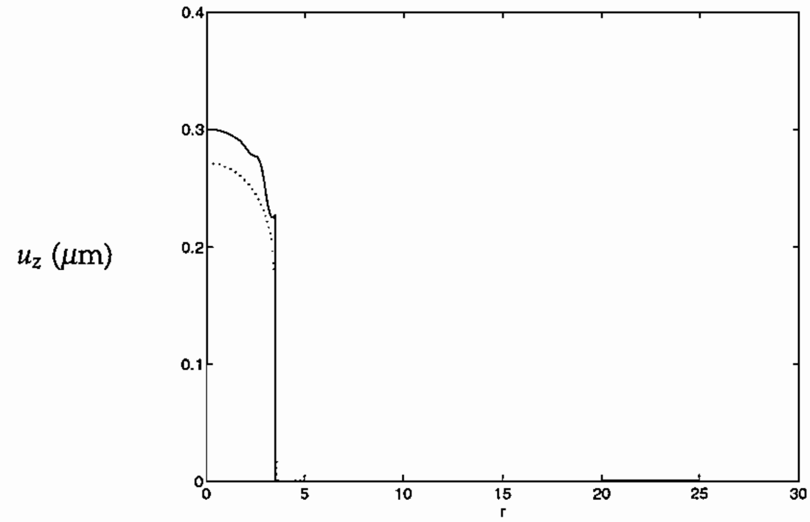

Figure 9 Axial displacement along the crack plane (radial direction).

9 CONCLUSIONS

A new method for modelling unidirectional composites has been devised by McCartney (1993). This method is a discretisation procedure which involves the representation of composite materials by a set of perfectly bonded concentric cylinders. The problem considered is where a fibre is embedded in a matrix material, which in turn is embedded in a composite material. This is thus a three-phase problem. The fibre is cracked into sections of equal length and one such section (comprising fibre, matrix and composite) is modelled in order to find various stress and displacement distributions within the section.

The analysis gives rise to systems of ordinary differential equations and appropriate boundary conditions, the solution of which is considered in this report. It has been demonstrated, by using the solution of the characteristic equation of the system, that eigensolutions may be found and that the computation of unknown coefficients may be carried out in a stable manner by giving careful consideration to the treatment of (possibly) large exponents arising from the characteristic equation solution. We have also shown that the size of the system of linear algebraic equations which is consistent with the prescribed boundary conditions may be considerably reduced.

The solution functions of the system of differential equations may be post-processed to recover the required stress and displacement fields and we have compared these results with those obtained from a finite element analysis. In general, there is good agreement between the two methods and in cases where there is not, it is believed that the results of the concentric cylinder method are more reliable.

10 ACKNOWLEDGEMENTS

The analytical approach used to formulate the coefficients of the systems of differential equations and corresponding boundary conditions, and the work described in this report, were carried out as part of the NPL Strategic Research programme. I would like to thank Dr L N McCartney for carrying out the work of the first part of this project, namely the analysis of the multiple cylinder method, Dr L N McCartney and Prof G T Symm for their valuable comments on earlier drafts of this report, and M Nedele of the University of Bristol for providing comparison results obtained by finite element analyses. I acknowledge that the actual problems solved in this report arose from participation in PREDICT. This project (Prediction of Damage Initiation and Growth in Composite Materials) is a collaboration between British Aerospace (Sowerby Research Centre), Ciba Geigy Plastics, Rolls Royce, Tenax Fibres, National Physical Laboratory, University of Bristol and the University of Surrey. The project is led by British Aerospace and is funded under the LINK Structural Composites programme of the DTI's Research and Technology Initiative (IED Grant Ref. RA 6/25/01).

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A INPUT DATA

The data required for equations (1) to (7) was as follows:

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Columns 1 through 7

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G:

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Columns 8 through 14

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4.9496e−2	5.1705e−2	5.8081e−2	9.1263e−2	1.4000e−1	1.1154e−1	6.0101e−2
3.8085e−2	3.8085e−2	3.8085e−2	3.8085e−2	8.1612e−3	5.9301e−2	7.6317e−2
6.0959e−2	6.0959e−2	6.0959e−2	6.0959e−2	6.2540e−4	1.9612e−2	1.0137e−1
8.1792e−2	8.1792e−2	8.1792e−2	8.1792e−2	8.3913e−4	8.3913e−4	2.5165e−2
9.5701e−2	9.5701e−2	9.5701e−2	9.5701e−2	9.8184e−4	9.8184e−4	9.8184e−4
9.4015e−2	9.4015e−2	9.4015e−2	9.4015e−2	9.6454e−4	9.6454e−4	9.6454e−4
9.2103e−2	9.2103e−2	9.2103e−2	9.2103e−2	9.4493e−4	9.4493e−4	9.4493e−4
3.9110e−1	3.9110e−1	3.9110e−1	3.9110e−1	4.0125e−3	4.0125e−3	4.0125e−3
5.5984e−1	5.5984e−1	5.5984e−1	5.5984e−1	5.7437e−3	5.7437e−3	5.7437e−3

Columns 15 through 20

1.2188e−2	1.2188e−2	1.2188e−2	1.2188e−2	1.2188e−2	1.2188e−2
1.8026e−2	1.8026e−2	1.8026e−2	1.8026e−2	1.8026e−2	1.8026e−2
1.8047e−2	1.8047e−2	1.8047e−2	1.8047e−2	1.8047e−2	1.8047e−2
1.8032e−2	1.8032e−2	1.8032e−2	1.8032e−2	1.8032e−2	1.8032e−2
1.8007e−2	1.8007e−2	1.8007e−2	1.8007e−2	1.8007e−2	1.8007e−2
1.7550e−2	1.7550e−2	1.7550e−2	1.7550e−2	1.7550e−2	1.7550e−2
9.1684e−3	9.1684e−3	9.1684e−3	9.1684e−3	9.1684e−3	9.1684e−3
2.1337e−1	2.6122e−1	3.0235e−1	3.3817e−1	4.7506e−1	5.7303e−1
8.4882e−2	8.4882e−2	8.4882e−2	8.4882e−2	8.4882e−2	8.4882e−2
1.9222e−1	1.9222e−1	1.9222e−1	1.9222e−1	1.9222e−1	1.9222e−1
2.0642e−1	2.0642e−1	2.0642e−1	2.0642e−1	2.0642e−1	2.0642e−1
−8.6309e−4	−6.2004e−2	−1.1456e−1	−1.6033e−1	−3.3523e−1	−4.6042e−1
7.6317e−2	7.6317e−2	7.6317e−2	7.6317e−2	7.6317e−2	7.6317e−2
1.2286e−1	1.2286e−1	1.2286e−1	1.2286e−1	1.2286e−1	1.2286e−1
1.3628e−1	1.6648e−1	1.6648e−1	1.6648e−1	1.6648e−1	1.6648e−1
3.6208e−2	1.6694e−1	1.9747e−1	1.9747e−1	1.9747e−1	1.9747e−1
9.6454e−4	3.5866e−2	1.6666e−1	1.9746e−1	1.9746e−1	1.9746e−1
9.4493e−4	9.4493e−4	3.5596e−2	1.6643e−1	1.9744e−1	1.9744e−1
4.0125e−3	4.0125e−3	4.0125e−3	3.8466e−2	7.2402e−1	9.3742e−1
5.7437e−3	5.7437e−3	5.7437e−3	5.7437e−3	4.3254e−1	1.7060e+0

H:

Columns 1 through 7

−2.2631e−2	−1.6973e−2	−1.6973e−2	−1.6973e−2	−1.6973e−2	−1.6973e−2	−1.6973e−2
1.1315e−2	−6.7892e−3	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	1.0184e−2	−7.2741e−3	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	9.6988e−3	−7.5435e−3	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	9.4294e−3	−7.7150e−3	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	9.2579e−3	−8.2778e−3	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	9.5884e−3	−1.6912e−1
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	1.7034e−1
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0

Columns 8 through 14

−1.6973e−2	−1.6973e−2	−1.6973e−2	−1.6973e−2	−1.6973e−2	−1.6973e−2	−1.6973e−2
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
−8.8334e+0	1.0073e−2	1.0073e−2	1.0073e−2	−1.0352e−4	−1.0352e−4	−1.0352e−4
8.9066e+0	−7.7653e−1	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	8.3711e−1	−5.2696e−1	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	5.8243e−1	−6.7122e−1	0.0000e+0	0.0000e+0	0.0000e+0
−1.3138e−2	−1.3138e−2	−1.3138e−2	7.0898e−1	−2.9298e−2	1.2925e−4	1.2925e−4
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	3.0935e−2	−1.2009e−2	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	1.3400e−2	−9.3637e−3
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	1.0600e−2
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0

Columns 15 through 20

$-1.6973e-2$	$-1.6973e-2$	$-1.6973e-2$	$-1.6973e-2$	$-1.6973e-2$	$-1.6973e-2$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$-1.0352e-4$	$-1.0352e-4$	$-1.0352e-4$	$-1.0352e-4$	$-1.0352e-4$	$-1.0352e-4$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$1.2925e-4$	$1.2925e-4$	$1.2925e-4$	$1.2925e-4$	$1.2925e-4$	$1.2925e-4$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$-6.4499e-3$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$7.5249e-3$	$-6.5216e-3$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$0.0000e+0$	$7.4533e-3$	$-6.5764e-3$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$
$0.0000e+0$	$0.0000e+0$	$7.3985e-3$	$-6.6197e-3$	$0.0000e+0$	$0.0000e+0$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$7.3552e-3$	$-4.6583e-4$	$0.0000e+0$
$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$0.0000e+0$	$9.3166e-4$	$-5.5899e-4$

Radii:

$5.0000e-1$
$1.0000e+0$
$1.5000e+0$
$2.0000e+0$
$2.5000e+0$
$3.0000e+0$
$3.4750e+0$
$3.5000e+0$
$3.5250e+0$
$3.8000e+0$
$4.2000e+0$
$4.5185e+0$
$4.7500e+0$
$5.3000e+0$
$6.0000e+0$
$7.0000e+0$
$8.0000e+0$
$9.0000e+0$
$1.0000e+1$
$2.0000e+1$
$3.0000e+1$

Sigma:

3.5051e+0
3.5051e+0
3.5051e+0
3.5051e+0
3.5051e+0
3.5051e+0
3.5051e+0
3.5051e+0
9.9152e−2
9.9152e−2
9.9152e−2
9.9152e−2
2.1451e+0
2.1451e+0
2.1451e+0
2.1451e+0
2.1451e+0
2.1451e+0
2.1451e+0
2.1451e+0
2.1451e+0

Nf = 8

Mua(9) = 1.41971

D:

Columns 1 through 7

4.4596e−3	3.3469e−3	2.2837e−3	1.7678e−3	1.4709e−3	1.2834e−3	1.1636e−3
6.6780e−3	1.1216e−2	9.1348e−3	7.0712e−3	5.8834e−3	5.1334e−3	4.6544e−3
6.8482e−3	1.3696e−2	1.8373e−2	1.5910e−2	1.3238e−2	1.1550e−2	1.0472e−2
7.0704e−3	1.4141e−2	2.1211e−2	2.6101e−2	2.3534e−2	2.0534e−2	1.8618e−2
7.3539e−3	1.4708e−2	2.2062e−2	2.9416e−2	3.4586e−2	3.2084e−2	2.9090e−2
7.5117e−3	1.5023e−2	2.2535e−2	3.0047e−2	3.7558e−2	4.2884e−2	4.0672e−2
4.0649e−3	8.1298e−3	1.2195e−2	1.6260e−2	2.0325e−2	2.4389e−2	2.6277e−2
4.0502e−4	8.1004e−4	1.2151e−3	1.6201e−3	2.0251e−3	2.4301e−3	2.8149e−3
2.3654e−3	4.7309e−3	7.0963e−3	9.4617e−3	1.1827e−2	1.4193e−2	1.6440e−2
5.1594e−3	1.0319e−2	1.5478e−2	2.0638e−2	2.5797e−2	3.0957e−2	3.5858e−2
5.2168e−3	1.0434e−2	1.5650e−2	2.0867e−2	2.6084e−2	3.1301e−2	3.6257e−2
3.7640e−3	7.5279e−3	1.1292e−2	1.5056e−2	1.8820e−2	2.2584e−2	2.6160e−2
5.1827e−3	1.0365e−2	1.5548e−2	2.0731e−2	2.5914e−2	3.1096e−2	3.6020e−2
8.2955e−3	1.6591e−2	2.4887e−2	3.3182e−2	4.1478e−2	4.9773e−2	5.7654e−2
1.1131e−2	2.2261e−2	3.3392e−2	4.4522e−2	5.5653e−2	6.6783e−2	7.7357e−2
1.3023e−2	2.6047e−2	3.9070e−2	5.2094e−2	6.5117e−2	7.8141e−2	9.0513e−2
1.2794e−2	2.5588e−2	3.8382e−2	5.1176e−2	6.3970e−2	7.6764e−2	8.8918e−2
1.2534e−2	2.5068e−2	3.7601e−2	5.0135e−2	6.2669e−2	7.5203e−2	8.7110e−2
5.3223e−2	1.0645e−1	1.5967e−1	2.1289e−1	2.6612e−1	3.1934e−1	3.6990e−1
7.6186e−2	1.5237e−1	2.2856e−1	3.0474e−1	3.8093e−1	4.5711e−1	5.2949e−1
1.5181e−2	3.0362e−2	4.5542e−2	6.0723e−2	7.5904e−2	9.1085e−2	1.0551e−1

Columns 8 through 14

$1.1584e-3$	$1.1460e-3$	$1.0183e-3$	$8.5705e-4$	$7.4478e-4$	$7.0675e-4$	$6.2942e-4$
$4.6337e-3$	$4.5840e-3$	$4.0733e-3$	$3.4282e-3$	$2.9791e-3$	$2.8270e-3$	$2.5177e-3$
$1.0426e-2$	$1.0314e-2$	$9.1650e-3$	$7.7134e-3$	$6.7030e-3$	$6.3607e-3$	$5.6647e-3$
$1.8535e-2$	$1.8336e-2$	$1.6293e-2$	$1.3713e-2$	$1.1916e-2$	$1.1308e-2$	$1.0071e-2$
$2.8961e-2$	$2.8650e-2$	$2.5458e-2$	$2.1426e-2$	$1.8619e-2$	$1.7669e-2$	$1.5735e-2$
$4.0491e-2$	$4.0057e-2$	$3.5594e-2$	$2.9957e-2$	$2.6033e-2$	$2.4704e-2$	$2.2000e-2$
$2.6166e-2$	$2.5885e-2$	$2.3002e-2$	$1.9359e-2$	$1.6823e-2$	$1.5964e-2$	$1.4217e-2$
$2.8297e-3$	$2.8222e-3$	$2.5078e-3$	$2.1106e-3$	$1.8341e-3$	$1.7405e-3$	$1.5500e-3$
$1.6558e-2$	$1.7164e-2$	$1.7736e-2$	$1.4927e-2$	$1.2971e-2$	$1.2309e-2$	$1.0962e-2$
$3.6116e-2$	$3.7487e-2$	$4.9292e-2$	$4.6463e-2$	$4.0376e-2$	$3.8315e-2$	$3.4122e-2$
$3.6518e-2$	$3.7904e-2$	$5.2668e-2$	$6.6900e-2$	$6.1395e-2$	$5.8260e-2$	$5.1885e-2$
$2.6348e-2$	$2.7348e-2$	$3.8000e-2$	$5.2536e-2$	$5.9725e-2$	$5.7385e-2$	$5.1106e-2$
$3.6279e-2$	$3.7656e-2$	$5.2323e-2$	$7.2338e-2$	$8.7401e-2$	$8.9464e-2$	$8.3427e-2$
$5.8069e-2$	$6.0273e-2$	$8.3750e-2$	$1.1578e-1$	$1.3990e-1$	$1.4439e-1$	$1.5132e-1$
$7.7914e-2$	$8.0872e-2$	$1.1237e-1$	$1.5535e-1$	$1.8770e-1$	$1.9374e-1$	$2.0869e-1$
$9.1164e-2$	$9.4625e-2$	$1.3148e-1$	$1.8177e-1$	$2.1963e-1$	$2.2669e-1$	$2.4418e-1$
$8.9558e-2$	$9.2958e-2$	$1.2916e-1$	$1.7857e-1$	$2.1576e-1$	$2.2269e-1$	$2.3988e-1$
$8.7737e-2$	$9.1068e-2$	$1.2654e-1$	$1.7494e-1$	$2.1137e-1$	$2.1817e-1$	$2.3500e-1$
$3.7256e-1$	$3.8671e-1$	$5.3733e-1$	$7.4286e-1$	$8.9755e-1$	$9.2641e-1$	$9.9789e-1$
$5.3330e-1$	$5.5355e-1$	$7.6915e-1$	$1.0634e+0$	$1.2848e+0$	$1.3261e+0$	$1.4284e+0$
$1.0627e-1$	$1.1030e-1$	$1.5326e-1$	$2.1189e-1$	$2.5601e-1$	$2.6424e-1$	$2.8463e-1$

Columns 15 through 21

$5.5094e-4$	$4.6513e-4$	$3.9981e-4$	$3.4816e-4$	$3.0608e-4$	$9.5649e-5$	$0.0000e+0$
$2.2038e-3$	$1.8605e-3$	$1.5993e-3$	$1.3927e-3$	$1.2243e-3$	$3.8260e-4$	$0.0000e+0$
$4.9585e-3$	$4.1862e-3$	$3.5983e-3$	$3.1335e-3$	$2.7547e-3$	$8.6084e-4$	$0.0000e+0$
$8.8150e-3$	$7.4421e-3$	$6.3970e-3$	$5.5706e-3$	$4.8972e-3$	$1.5304e-3$	$0.0000e+0$
$1.3774e-2$	$1.1628e-2$	$9.9954e-3$	$8.7041e-3$	$7.6520e-3$	$2.3912e-3$	$0.0000e+0$
$1.9257e-2$	$1.6258e-2$	$1.3975e-2$	$1.2170e-2$	$1.0699e-2$	$3.3433e-3$	$0.0000e+0$
$1.2444e-2$	$1.0506e-2$	$9.0308e-3$	$7.8642e-3$	$6.9135e-3$	$2.1605e-3$	$0.0000e+0$
$1.3568e-3$	$1.1454e-3$	$9.8460e-4$	$8.5740e-4$	$7.5376e-4$	$2.3555e-4$	$0.0000e+0$
$9.5955e-3$	$8.1009e-3$	$6.9634e-3$	$6.0638e-3$	$5.3308e-3$	$1.6659e-3$	$0.0000e+0$
$2.9868e-2$	$2.5216e-2$	$2.1675e-2$	$1.8875e-2$	$1.6593e-2$	$5.1854e-3$	$0.0000e+0$
$4.5416e-2$	$3.8342e-2$	$3.2958e-2$	$2.8700e-2$	$2.5231e-2$	$7.8848e-3$	$0.0000e+0$
$4.4734e-2$	$3.7766e-2$	$3.2463e-2$	$2.8269e-2$	$2.4852e-2$	$7.7663e-3$	$0.0000e+0$
$7.3026e-2$	$6.1652e-2$	$5.2994e-2$	$4.6148e-2$	$4.0570e-2$	$1.2678e-2$	$0.0000e+0$
$1.3843e-1$	$1.1687e-1$	$1.0046e-1$	$8.7478e-2$	$7.6904e-2$	$2.4033e-2$	$0.0000e+0$
$2.2183e-1$	$1.9903e-1$	$1.7108e-1$	$1.4898e-1$	$1.3097e-1$	$4.0929e-2$	$0.0000e+0$
$2.6754e-1$	$2.8843e-1$	$2.5990e-1$	$2.2632e-1$	$1.9897e-1$	$6.2177e-2$	$0.0000e+0$
$2.6282e-1$	$2.9703e-1$	$3.1841e-1$	$2.8940e-1$	$2.5442e-1$	$7.9507e-2$	$0.0000e+0$
$2.5748e-1$	$2.9099e-1$	$3.2558e-1$	$3.4697e-1$	$3.1727e-1$	$9.9147e-2$	$0.0000e+0$
$1.0933e+0$	$1.2356e+0$	$1.3825e+0$	$1.5325e+0$	$1.6707e+0$	$9.4782e-1$	$0.0000e+0$
$1.5651e+0$	$1.7687e+0$	$1.9790e+0$	$2.1937e+0$	$2.4114e+0$	$3.2942e+0$	$0.0000e+0$
$3.1185e-1$	$3.5244e-1$	$3.9434e-1$	$4.3711e-1$	$4.8050e-1$	$9.2787e-1$	$0.0000e+0$

E:

Columns 1 through 7

1.1489e − 3	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	1.1489e − 3	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	1.1489e − 3	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	1.1489e − 3	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	1.1489e − 3	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	1.1489e − 3	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	1.1489e − 3
1.4263e − 3	2.8526e − 3	4.2789e − 3	5.7052e − 3	7.1315e − 3	8.5578e − 3	9.9128e − 3
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
−1.7879e − 3	−3.5758e − 3	−5.3637e − 3	−7.1516e − 3	−8.9395e − 3	−1.0727e − 2	−1.2426e − 2
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
−2.4760e − 5	−4.9521e − 5	−7.4281e − 5	−9.9041e − 5	−1.2380e − 4	−1.4856e − 4	−1.7208e − 4

Columns 8 through 14

0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
1.1133e − 2	−1.1790e − 1	−1.0476e − 1	−8.8171e − 2	−7.6621e − 2	−7.2708e − 2	−6.4753e − 2
0.0000e + 0	1.3031e − 1	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	1.3031e − 1	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	1.3031e − 1	0.0000e + 0	0.0000e + 0	0.0000e + 0
−1.2515e − 2	−1.2991e − 2	−1.8050e − 2	−2.4955e − 2	1.0016e − 1	9.2903e − 2	8.2738e − 2
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	2.2551e − 3	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	2.2551e − 3
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0	0.0000e + 0
−1.7332e − 4	−1.7990e − 4	−2.4997e − 4	−3.4559e − 4	−4.1756e − 4	−4.3098e − 4	−4.6423e − 4

Columns 15 through 21

0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
−5.6679e−2	−4.7851e−2	−4.1132e−2	−3.5818e−2	−3.1489e−2	−9.8402e−3	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
7.2422e−2	6.1142e−2	5.2556e−2	4.5767e−2	4.0234e−2	1.2573e−2	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
2.2551e−3	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	2.2551e−3	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	2.2551e−3	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	2.2551e−3	0.0000e+0	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	2.2551e−3	0.0000e+0	0.0000e+0
0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	0.0000e+0	2.2551e−3	0.0000e+0
−5.0864e−4	−5.7484e−4	−6.4317e−4	−7.1294e−4	−7.8371e−4	−1.5134e−3	0.0000e+0